$$\begin{bmatrix} 0.00004 \\ 0.00005 \end{bmatrix} \quad \therefore \text{ Soln } = \begin{bmatrix} 0.4966 \\ 0.3699 \end{bmatrix}$$

<u>Modified Newton's Method</u>: An essential inconvenience in forming the Newton's process

$$X^{(p+1)} = X^{(p)} - W^{-1}(X^{(p)})\, f(X^{(p)}) \quad \text{--- (1)}$$

is the neccessity to compute the inverse matrix $W^{-1}(X^{(p)})$ at each step. If the matrix $W^{-1}(X)$ is continuous in the nbd. of the desired soln. X^* & the initial approximation $X^{(0)}$ is sufficiently close to X^*, then we can approximately put $W^{-1}(X^{(p)}) \simeq W^{-1}(X^{(0)})$.

Thus we have the modified process as

$$\xi^{(p+1)} = \xi^{(p)} - W^{-1}(X^{(0)})\, f(X^{(p)}) \quad \text{--- (2)}$$

$p = 0, 1, 2, \ldots$

$\xi^{(0)} = X^{(0)}$ & $\xi^{(1)} = X^{(1)}$

Ex: i) $x^2 + y^2 = 1$, $x^3 - y = 0$, | $x = 0.9$, $y = 0.5$

ii) $y \cdot \cos(xy) + 1 = 0$

$\sin(xy) + x - y = 0$ | $x = 1$, $y = 2$

iii) $x^3 + y^3 = 53$

$2y^3 + z^4 = 69$

$3x^5 + 10z^2 = 770$ | $x = 3$, $y = 3$, $z = 2$

iv) $x = \log(y/x) + 1$ | $x = 1$

$y = 0.4 + z^2 - 2x^2$ | $y = 2.2$

$z = 2 + x^4/20$ | $z = 2$

v) $10x + \sin(x+y) = 1$ | $x = 1/10$

$8y - \cos^2(z-y) = 1$ | $y = 1/4$

$12z + \sin(z) = 1$ | $z = 1/12$

Complex Root of Non Linear Eqn:

$f(z) = 0$ --- (1) $\quad z = x + iy$

$$\left.\begin{array}{l} u(x,y) = 0 \\ v(x,y) = 0 \end{array}\right\} \text{--- (2)} \qquad u(x,y) + i\,v(x,y) = 0$$

let $x_{k+1} = x_k + \Delta x_k$ & $y_{k+1} = y_k + \Delta y_k$

Using previous methods of Taylor series we get

$$\Delta x_k \cdot u_x(x_k, y_k) + \Delta y_k \cdot u_y(x_k, y_k) = -u(x_k, y_k)$$

$$\Delta x_k \cdot v_x(x_k, y_k) + \Delta y_k \cdot v_y(x_k, y_k) = -v(x_k, y_k)$$

$$\begin{bmatrix} \Delta x_k \\ \Delta y_k \end{bmatrix} = - J^{-1}(x_k, y_k) \begin{bmatrix} u \\ v \end{bmatrix}_{(x_k, y_k)}$$

$$\therefore \begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} - W^{-1}(x^{(k)}, y^{(k)}) \begin{bmatrix} f \end{bmatrix}$$

$$x_{k+1} = x_k - J^{-1}\{u(x_k, y_k)\, v_y(x_k, y_k) - v(x_k, y_k)\, u_y(x_k, y_k)\}$$

$$y_{k+1} = y_k - J^{-1}\{u_x(x_k, y_k)\, v(x_k, y_k) - u(x_k, y_k)\, v_x(x_k, y_k)\}$$

$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

Using Cramer's rule, $u_x = v_y$ & $v_x = -u_y$

$$J = u_x^2 + v_x^2$$

Alternatively we may apply Newton - Rhapson method directly to solve $f(x) = 0$ to the eqn. $f(z) = 0$ in the form -

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)} \quad \text{------} \quad (A)$$

$$k = 0, 1, 2, \ldots$$

The initial approximation z_0 must also be complex ie, $z_0 = x_0 + iy_0$

After one iteration z_1 is obtained & NR. method should be applied to deflated polynomial ie,

$$f^*(z) = \frac{f(z)}{z - z_1}$$

This process can be applied on each iteration

If k roots have been obtained then

$$f^*(z) = \frac{f(z)}{(z-z_1)(z-z_2)\cdots(z-z_k)} \quad \text{for next iteration}$$

$$\therefore z_{k+1} = z_k - \frac{f^*(z_k)}{f'^*(z_k)}$$

$$\frac{f^{*\prime}}{f^*} = \frac{d}{dz}\left(\log f^*\right) = \frac{d}{dz}\left\{\log\left(\frac{f(z)}{z-z_1}\right)\right\}$$

$$= \frac{d}{dz}\left\{\log f(z) - \log(z-z_1)\right\}$$

$$= \frac{f'}{f} - \frac{1}{z-z_1}$$

Any zeroes obtained using deflated polynomial should be refined using Newton's method to the original polynomial with this root as the initial / starting approximation.

The zeroes should be computed in the increasing order of magnitude, only then we can have correct result.

Q. Find the complex root by N.R. method

i) $z^2+1=0$; $z_0 = (1+i)/2$

ii) $z^3 - 4i\,z^2 - 3e^z = 0$; $z_0 = -0.53 - 0.36i$

iii) $z^3+1=0$; $z_0 = 0.25 + 0.25i$

iv) $e^z - 0.2z + 1 = 0$; $z_0 = \pi i = 3.142i$

Soln i) $z = x+iy$ $\therefore$ $z^2+1=0$

$\Rightarrow (x+iy)^2+1=0$

$\Rightarrow (x^2-y^2+1) + 2xy\,i = 0$

$$\Rightarrow \left.\begin{aligned} u(x,y) &= x^2-y^2+1 = 0 \\ v(x,y) &= 2xy = 0 \end{aligned}\right] \quad \text{--- (1)}$$

$\therefore\ u_x = 2x \qquad u_y = -2y$

$v_x = 2y \qquad v_y = 2x$ $\Big\}$; $u_x = v_y$

& $v_x = -u_y$

(Analytic)

$$\therefore\ J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x^2 + v_x^2$$

$$\Delta x = \begin{vmatrix} u & u_y \\ v & v_y \end{vmatrix} = u u_x + v v_x$$

$$\Delta y = v u_x - u v_x$$

$$\left.\begin{aligned} x_{k+1} &= x_k - \left[\frac{u u_x + v v_x}{u_x^2 + v_x^2}\right]_{(x_k, y_k)} \\ \& \ y_{k+1} &= y_k - \left[\frac{v u_x - u v_x}{u_x^2 + v_x^2}\right]_{(x_k, y_k)} \end{aligned}\right\} k = 0, 1, 2, \ldots$$

Now,

$$x_1 = x_0 - \left[\frac{u u_x + v v_x}{u_x^2 + v_x^2}\right]_{(x_0, y_0)} \doteq -0.25$$

$$y_1 = \ldots = -0.75$$ Continue ...

---

iii) $z^3 + 1 = 0 \qquad z_0 = 0.25 + 0.25i$

$f(z) = z^3 + 1 \qquad f'(z) = 3z^2$

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)} \qquad k = 0, 1, 2, \ldots$$

$$= z_k - \frac{z_k^3 + 1}{3z_k^2}$$

$\therefore\ z_1 = \ldots = 0.16667 + 2.83333i$

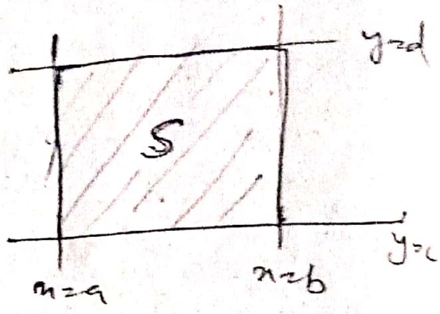
$z_2 = \cancel{0.0522} \ldots \quad z_8 = 0.5 + 0.866025403\,i$

Evaluation of Double & Triple Integrals

Let $I = \iint_S f(x,y)\, dx\, dy$ where,

S is the rectangular region $a \le x \le b$, $c \le y \le d$

The evaluation of I can be reduced to the repeated iteration of single integration,



$$\therefore I = \int_a^b \int_c^d f(x,y)\, dx\, dy = \int_a^b dx \int_c^d f(x,y)\, dy \quad \text{--- (1)}$$

Let
$$F(x) = \int_c^d f(x,y)\, dy \quad \text{---- (2)}$$

$$\therefore I = \int_a^b F(x)\, dx \quad \text{--- (3)}$$

Using trapezoidal rule for integration,

$$I_T = \frac{h}{2}\left[F_0 + 2(F_1 + F_2 + \cdots + F_{n-1}) + F_n\right] \quad \text{--- (4)}$$

where, $F_i = F(x_i) = \int_c^d f(x_i, y)\, dy$ --- (5)

Using Simpson's 1/3rd rule on (3) we have,

$$I_{S1/3} = \frac{h}{3}\left[F_0 + 4(F_1 + F_3 + F_5 + \cdots + F_{n-1}) + 2(F_2 + F_4 + \cdots + F_{n-2}) + F_n\right] \quad \text{--- (6)}$$

$h = \frac{b-a}{n}$.

$\therefore$ Evaluation of integral I is reduced to evaluation of $n+1$ integrals of the form

$$F_i = \int_c^d f(x_i, y)\, dy .$$

which can again be evaluated by trapezoidal or simpson's methods.

Using trapezoidal rule for F_i evaluation

$$F_i = \frac{k}{2}\left[f(x_i, y_0) + 2\{ f(x_i, y_1) + f(x_i, y_2) + \cdots + f(x_i, y_{m-1})\} + f(x_i, y_m) \right]$$

where, $k = \dfrac{d-c}{m}$

or $$F_i = \frac{k}{2}\left[f_{i,0} + 2(f_{i,1} + f_{i,2} + \cdots + f_{i,m-1}) + f_{i,m} \right] \quad \cdots (7)$$

Also, using simpson's method,

$$F_i = \frac{k}{3}\left[f_{i,0} + 4(f_{i,1} + f_{i,3} + \cdots + f_{i,m-1}) + 2(f_{i,2} + f_{i,4} + \cdots + f_{i,m-2}) + f_{i,m} \right] \quad (8)$$

Q. Evaluate $\iint_S (x^2+y^2)\, dx\, dy$ for $S: 1 \leq x \leq 3,\ 1 \leq y \leq 2$ using trapezoidal & simpsons methods & compare with actual value.

Soln: i) Trapezoidal Rule.

$$I = \int_1^3 dx \int_1^2 (x^2+y^2)\, dy \quad \text{--- (1)}$$

Let $$F(x) = \int_1^2 (x^2+y^2)\, dy \quad \text{--- (2)}$$

$$\therefore I_T = \int_1^3 F(x)\, dx$$

Let $h = \frac{3-1}{4}$

$= 0.5$

$$= \frac{h}{2}\left[F_0 + 2(F_1+F_2+F_3)+F_4\right]$$

x_0 x_1 x_2 x_3 x_4

1 1.5 2 2.5 3

where, $$F_i = \int_1^2 (x_i^2+y^2)\, dy$$

Let $k = \frac{2-1}{2} = 0.5$

$$\therefore F_0 = \frac{k}{2}\left[(x_0^2+y_0^2) + 2(x_0^2+y_1^2)+(x_0^2+y_2^2)\right]$$

y_0 y_1 y_2

1 1.5 2

$= \cdots = 3.375$

$$F_1 = \frac{k}{2}\left[(x_1^2+y_0^2) + 2(x_1^2+y_1^2) + (x_1^2+y_2^2)\right]$$

$= \cdots = 4.625$

$F_2 = 6.375$, $F_3 = 8.625$, $F_4 = 11.375$

$\therefore I_T = \cdots = 13.5$

Exact value $I = \cdots = 13.3333$

i) Simpson's $\frac{1}{3}$rd rule.

$$I_{S1/3} = \frac{h}{3}\left[F_0 + 4(F_1 + F_3) + 2F_2 + F_4\right]$$

$h = 0.5$

$$F_i = \int_1^2 f(x_i, y)\,dy$$

$\therefore F_0 = 3.33332$ $\quad F_1 = 4.583337$

$F_2 = 6.3333$ $\quad F_3 = 8.583331$ $\quad F_4 = 11.33338$

$\therefore I_{S1/3} = \cdots = 13.3333 = I$

Ex:

$$\iint_S (x^2+2y)\,dx.dy \qquad S: 0 \le x \le 2,\ 0 \le y \le 1$$

i) $n_x = 4$, $n_y = 2$ $\qquad$ ii) $n_x = 2$, $n_y = 2$

Q $I = \int_0^{\pi/4}\int_0^{\pi/4} \sin(x+y)\,dx\,dy$ $\qquad n_x = 4$, $n_y = 2$, $S\frac{1}{3}$

Soln: $I_{S1/3} = \int_0^{\pi/4} dx \int_0^{\pi/4} \sin(x+y)\,dy = \int_0^{\pi/4} F(x)\,dx$

$= \frac{h}{3}\left[F_0 + 4(F_1 + F_3) + 2F_2 + F_4\right]$

$F_i = F(x_i) = \int_0^{\pi/4} \sin(x_i + y)\,dy$

$F_0 = F(x_0) = \frac{k}{3}\left[\sin(x_0+y_0) + 4\sin(x_0+y_1) + \sin(x_0+y_2)\right]$

$= 0.2929326$.

$h = \frac{\pi/4 - 0}{4} = \pi/16$

$k = \frac{\pi/4 - 0}{2} = \pi/8$

$F_2 = 0.54126\ldots$

Q $F_1 = 0.4252722$ $F_4 = 0.7072019$

$F_3 = 0.6364649$

$\therefore I_{S1/3} = \ldots\ldots = 0.4142727 \simeq 0.4143$

<u>Exact value</u> $I = \int_0^{\pi/4}\int_0^{\pi/4} \sin(x+y)\, dy\, dx$

$$I = \int_0^{\pi/4} \left[-\cos(x+y)\right]_0^{\pi/4} dx = \int_0^{\pi/4} \left[\cos x - \cos(x+\pi/4)\right] dx$$

$$= \ldots = 0.4142135$$

$\therefore$ Error $= I_{S1/3} - I = 0.0000591$

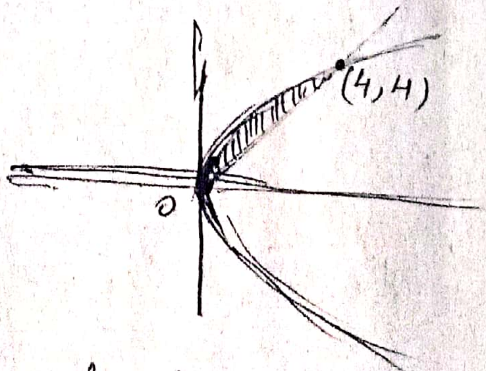
<u>Double Integral with Variable Limit</u>

Q Find the mass of plane lamina bounded by the curves $y^2 = 4x$ & $y = x$, given the density ρ at any point $P(x, y)$ is given by $\rho = x^2 + y^2$

Sol:

$$\therefore M = \iint_S \rho \cdot dx\, dy = \iint_S (x^2+y^2)\, dx\, dy$$

$$= \int_0^4 dx \int_x^{2\sqrt{x}} (x^2+y^2)\, dy$$



$$= \int_0^4 F(x)\, dx$$

$h = \frac{4-0}{4} = 1$

$$= \frac{h}{3}\left[F_0 + 4(F_1+F_3) + 2F_2 + F_4\right]$$

$$F_i = F(x_i) = \int_{x_i}^{2\sqrt{x_i}} (x_i^2 + y^2)\, dy$$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$0 \quad 1 \quad 2 \quad 3 \quad 4$

$$F_0 = \int_0^0 \cdots = 0$$

$$F_1 = \int_{x_1}^{2\sqrt{x_1}} (x_1^2 + y^2)\, dy = \int_1^2 (1 + y^2)\, dy \qquad k = \frac{2-1}{2} = 0.5$$

$$= \frac{k}{3}\left[(1^2 + y_0^2) + 4(1 + y_1^2) + (1 + y_2^2)\right] = 10/3$$

$$F_2 = \int_{x_2}^{2\sqrt{x_2}} (x_2^2 + y^2)\, dy = \int_2^{2\sqrt{2}} (2^2 + y^2)\, dy \qquad k = \frac{2\sqrt{2} - 2}{2} = \sqrt{2} - 1 \simeq 0.414$$

$$= \frac{0.414}{3}\left[(2^2 + 2^2) + 4(2^2 + 2.414^2) + 2^2 + (2\sqrt{2})^2\right]$$

$$= \frac{24.554167}{3} \qquad k = \frac{2\sqrt{3} - 3}{2} \simeq 0.232$$

$$F_3 = \int_{x_3}^{2\sqrt{x_3}} (x_3^2 + y^2)\, dy = \cdots = \frac{27.093723}{3}$$

$$F_4 = 0 \qquad \therefore M = \cdots = 21.94$$

Exact value $= \cdots = 22$

Q. Find mass of plane lamine bounded by $y^2 = 4x$ & $x^2 = 4y$ & $\rho = x^2 + 2y$.

Tripple Integrals

Ex : $I = \int_{z=0}^{1}\int_{y=0}^{2}\int_{x=0}^{4}(x+y+z)\,dx\,dy\,dz$ ---- ①

Soln: $F(y,z) = \int_{0}^{4}(x+y+z)\,dx$ --- ②

$\therefore I = \int_{z=0}^{1}\int_{y=0}^{2} F(y,z)\cdot dy\,dz$ --- ③

$\phi(z) = \int_{0}^{2} F(y,z)\,dy$ --- ④

$\therefore I = \int_{0}^{1}\phi(z)\,dz$ --- ⑤

$= h/3\,[\phi_0 + 4\phi_1 + \phi_2]$ $\quad h = \frac{1-0}{2} = 0.5$

where, $\phi_i = \int_{0}^{2} F(y, z_i)\,dy$ $\quad k = \frac{2-0}{2} = 1$

$\therefore$ $= k/3\,[F(y_0, z_i) + 4F(y_1, z_i) + F(y_2, z_i)]$

$\phi_0 = k/3\,[F(y_0, z_0) + 4F(y_1, z_0) + F(y_2, z_0)]$

$= 1/3\,[F(0,0) + 4F(1,0) + F(2,0)]$

Now $F(y,z) = \int_{0}^{4}(x+y+z)\,dx$ $\quad l = \frac{4-0}{2} = 2$

$\therefore F(0,0) = \int_{0}^{4} x\,dx$

$= \frac{l}{3}[x_0 + 4x_1 + x_2] = 8$

$F(1,0) = \frac{l}{3}\left[(x_0+1) + 4(x_1+1) + (x_2+1)\right] = 12$

$F(2,0) = \frac{l}{3}\left[(x_0+2) + 4(x_1+2) + (x_2+2)\right] = 16$

$\therefore \phi_0 = 24$

Now for $z = 0.5$

$F(0, 0.5) = \cdots = F(0,0) + 0.5 \times 2 = 10$

$F(1, 0.5) = 12 + 2 = 14$

$F(2, 0.5) = 16 + 2 = 16 + 2 = 18$

$\therefore \phi_1 = \phi_0 + \frac{1}{3}(2 + 4 \times 2 + 2) = 28$

$\phi_2 = \phi_1 + 4 = 32$

$\therefore I_{1/3} = \frac{0.5}{3}\left[24 + 4 \times 28 + 32\right] = 28$

Q2 $\int_0^1 \int_0^2 \int_0^4 (x^2 + y^2 + z^2)\, dx\, dy\, dz$

Numerical Soln. of Initial Value Problem:

$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dy^n}\right) = 0$

1) Picard's Method: Consider ODE (1st order)

$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x_0) = y_0$

$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\, dx \Rightarrow y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\, dx$

$\Rightarrow y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\, dx$

First approximation is given by -

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

Second approx., $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$

In general, $y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx$

Ex. $\frac{dy}{dx} = x+y \qquad y(0) = 1$

Soln:

$$y^{(1)} = 1 + \int_{0}^{x} (x+y_0)\, dx = 1 + \int_0^x (x+1)\, dx$$

$$= 1 + \left[\frac{(x+1)^2}{2}\right]_0^x = 1 + \left[\frac{(x+1)^2}{2} - \frac{1}{2}\right]$$

$$= \frac{(x+1)^2}{2} + \frac{1}{2} = 1 + x + \frac{x^2}{2}$$

$$y^{(2)} = 1 + \int_0^x \left[x + \frac{(x+1)^2}{2} + \frac{1}{2}\right] dx$$

$$= 1 + \left.\frac{x^2}{2}\right]_0^x + \frac{1}{2}\left[\frac{(x+1)^3}{3}\right]_0^x + \frac{1}{2}x$$

$$= 1 + \frac{x^2}{2} + \frac{1}{2}\left[\frac{(x+1)^3}{3} - \frac{1}{3}\right] + \frac{x}{2}$$

$$= 1 + x + \text{---}$$

$$y^{(5)} = \text{---} = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

$\frac{dy}{dx} = x+y$

3.43

1) Taylor's Series Method

$$y_0' = \frac{dy}{dx} = f(x, y) \quad ; \quad y(x_0) = y_0 \quad \cdots (1)$$

diff. above eqn we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \quad \cdots (2)$$

Expand $y(x)$ as a power series of $(x-x_0)$ in the nbd. of x_0 using Taylor series we get

$$y(x) = y_0 + (x-x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

Q Find by Taylor Series method the value of y at $x = 0.1$ & $x = 0.2$ to five place of decimal. $\frac{dy}{dx} = x^2 y - 1$ $y(0) = 1$

Soln: $y(0.1) = 0.9003$ $y(0.2) = \cancel{0.800}$ 0.8022

---

3) Euler's Method : Consider the eqn.

$$\frac{dy}{dx} = f(x, y) \quad ; \quad y(x_0) = y_0 \quad \cdots (1)$$

Tangent at $P_0(x_0, y_0)$ is

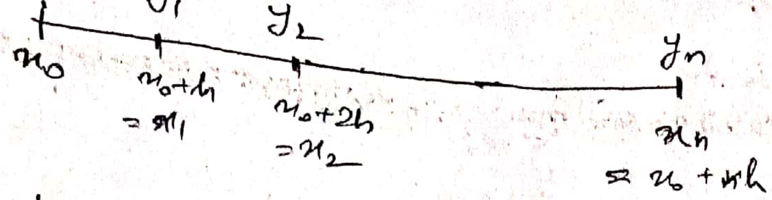
$$y - y_0 = \left(\frac{dy}{dx}\right)_{P_0} (x - x_0)$$

or, $$y = y_0 + f(x_0, y_0)(x - x_0) \quad \cdots (2)$$

$\Rightarrow$ $y_1 = y_0 + h f(x_0, y_0)$ & $y_2 = y_1 + h f(x_1, y_1)$

The general formula for Euler's method is

$$y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})$$



h is step length.

We use this method to find $\approx y(x_n)$

Lab

Euler's Modified Method:

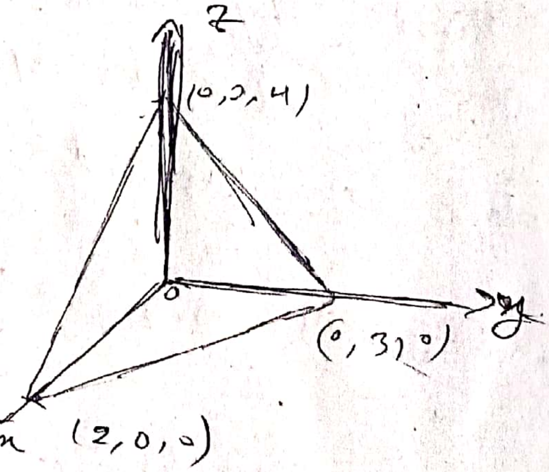
$$y_n = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)\right]$$

given by prev. meth.

Ex. Using S 1/3 find volume of solid bounded by the planes $x=0$, $y=0$, $z=0$ &

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$$

Sol. Method I:



$$V = \iint_A z\, dx\, dy$$

A: $x=0$, $y=0$ &

$$\frac{x}{2} + \frac{y}{3} = 1$$

$\therefore$ $x : 0 \to 2$ & $y : 0 \to 3(1 - x/2)$

& $z = 4(1 - x/2 - y/3)$

$$\therefore V = \int_0^2 \int_0^{3(1-x/2)} 4(1 - x/2 - y/3)\, dy\, dx$$

$$= \int_0^2 F(x)\, dx$$

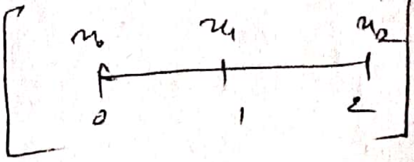
where,

$$F(x) = \int_0^{3(1-x/2)} 4(1 - x/2 - y/3)\, dy$$

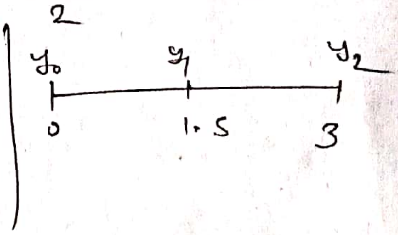
$\therefore V = \frac{h}{3}\left[F_0 + 4F_1 + F_2 \right]$, $h = \frac{2-0}{2} = 1$

$F_i = F(x_i) = \int_0^{3(1-x_i/2)} 4(1 - x_i/2 - y/3)\, dy$

$\therefore F_0 = \int_0^{3(1-x_0/2)} 4(1 - \frac{x_0}{2} - \frac{y}{3})\, dy$



$= \int_0^{3} 4(1 - y/3)\, dy$ $\quad k = \frac{3-0}{2} = 1.5$



$= \frac{k}{3} \cdot 4\left[(1 - \frac{y_0}{3}) + 4(1 - \frac{y_1}{3}) + (1 - \frac{y_2}{3}) \right]$

$= \ldots\ldots = 6$

$F_1 = 4\int_0^{3/2} (1 - 1/2 - y/3)\, dy = \int_0^{1.5} (0.5 - y/3)\, dy$

$= \ldots\ldots = 1.5$

$F_2 = \int_0^0 \ldots = 0$

$\therefore V = \frac{1}{3}(6 + 4\times 1.5 + 0) = 4$ cubic units

<u>Method II</u> : $V = \int_0^2 \int_0^{3(1-x/2)} 4(1 - x/2 - y/3)\, dy.dx$

D: $x = 0$ & $y = 0$ & $x/2 + y/3 = 1$

Let $X = x/2$, $Y = y/3$ | Transform

$\therefore x = 2X$ $\quad y = 3Y$

$J = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$

$$\therefore V = 4 \cdot \int_0^1 \int_0^{1-x} (1-x-y) \cdot |J| \cdot dY \cdot dX$$

$$= 24 \cdot \int_0^1 \int_0^{1-x} (1-x-y) \cdot dY \cdot dX = \text{------}$$

Method III : $$V = \iiint_D dx\, dy\, dz$$

$D:$ $x=0$, $y=0$, $z=0$ & $x/2 + y/3 + z/4 = 1$

$z = 0$ to $4(1 - x/2 - y/3)$

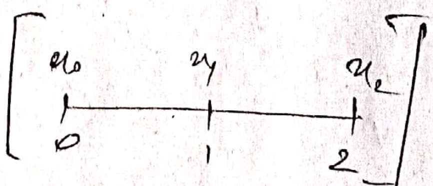
$y = 0$ to $3(1 - x/2)$

$x = 0$ to 2

$$\therefore F(x,y) = \int_0^{4(1-x/2-y/3)} dz$$

$$\therefore V = \int_0^2 \int_0^{3(1-x/2)} F(x,y)\, dy\, dx = \int_0^2 f(x)\, dx$$

$$\therefore V = \frac{h}{3}\left[f_0 + 4f_1 + f_2\right]$$



$$f_i = f(x_i) = \int_0^{3(1-x_i/2)} F(x_i, y)\, dy \qquad h = \frac{2-0}{2} = 1$$

$$= \frac{k}{3}\left[F(x_i, y_0) + 4F(x_i, y_1) + F(x_i, y_2)\right] \qquad k = \frac{3(1-x_i/2) - 0}{2}$$

$$\therefore f_0 = \int_0^{3(1-x_0/2)} F(x_0, y)\, dy = \int_0^3 F(0, y)\, dy$$

$$= \frac{1.5}{3}\left[F(0,y_0) + 4 \cdot F(0,y_1) + F(0,y_2)\right] \qquad k = \frac{3-0}{2} = 1.5$$

$$\therefore f_0 = \frac{1.5}{3}\left[F(0,0) + 4F(0,1.5) + F(0,3)\right]$$

$$F(x,y) = \int_0^{4(1-x/2-y/3)} dz = \int_0^{4(1-x/2-y/3)} z^0 \cdot dz$$

$$\therefore F(0,0) = \int_0^4 z^0 \cdot dz = \frac{2}{3}\left[1+4+1\right] = 4$$

$$F(0,1.5) = \int_0^2 z^0 \cdot dz = \frac{1}{3}\left[1+4+1\right] = 2$$

$$F(0,3) = \int_0^0 \ldots = 0$$

$$\therefore f_0 = \frac{1}{2}\left[4 + 4\times 2 + 0\right] = 6$$

Now

$$f_1 = \ldots = \frac{0.75}{3}\left[F(1,0) + 4F(1,0.75) + F(1,1.5)\right]$$

$$F(1,0) = \int_0^{4(1-1/2-0)} dz = \int_0^2 dz = 2 \quad \text{(from above)}$$

$$F(1,0.75) = \int_0^1 dz = \ldots = 1$$

$$F(1,1.5) = \ldots = \int_0^0 \ldots = 0$$

$$\therefore f_1 = \frac{0.75}{3}\left[2+4+0\right] = 6/4 = 3/2$$

$$f_2 = \int_0^0 \ldots = 0$$

$$\therefore V = \frac{1}{3}\left[6 + 4\times 3/2 + 0\right] = 4$$

Method IV : Transform to remove fractions.

$$V = \iiint_D dx\, dy\, dz$$

D: $x=0$, $y=0$, $z=0$ & $x/2 + y/3 + z/4 = 1$

Let $X = x/2$, $Y = y/3$, $Z = z/4$

$\therefore x = 2X$, $y = 3Y$, $z = 4Z$

$$\therefore J = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 24$$

$$\therefore V = \iiint_{D'} |J| \, dX\, dY\, dZ$$

D': $X=0$, $Y=0$, $Z=0$ & $X+Y+Z=1$

$$\therefore V = 24 \int_0^1 \int_0^{1-X} \int_0^{1-X-Y} dZ \cdot dY \cdot dX = \ldots$$

Q Using S1/3 find volume of $x^2+y^2+z^2=4$.

Soln: $V = 8 \iint_D z \, dx\, dy$ $\qquad z = (4-x^2-y^2)^{1/2}$

D: $x=0$, $y=0$, $x^2+y^2=4$

$$\therefore V = 8 \int_0^2 \int_0^{\sqrt{4-x^2}} (4-x^2-y^2)^{1/2} \, dy \cdot dx$$

$$\therefore V = 8 \int_0^2 F(x)\, dx \qquad F(x) = \int_0^{\sqrt{4-x^2}} (4-x^2-y^2)\, dy$$

$$V = 8 \times \frac{h}{3}\left[F_0 + 4F_1 + F_2\right]$$

$x_0 \quad x_1 \quad x_2$ (0, 1, 2)

$$f_0 = F(x_0) = \int_0^{\sqrt{4-x_0^2}} (4 - x_0^2 - y^2)\,dy$$

$$= \ldots = \ldots = \frac{2}{3}(1 + 2\sqrt{3})$$

$F_1 = \frac{1}{2}(1 + 2\sqrt{3}) \qquad F_2 = 0$

$\therefore V = \ldots = \underline{31.7447}$

But, Actual volume $= \frac{4}{3}\pi r^3 = \underline{33.51032146}$

Alternatively, Using Polar co-ordinates,

$D:\ x = 0,\ y = 0,\ x^2 + y^2 = 4$

Pt $x = r\cos\theta$ & $y = r\sin\theta$

$|J| = r = 2 \qquad r : 0 \to 2$

$\qquad\qquad\qquad \theta : 0 \to \pi/2$

$$\therefore V = 8 \iint_D (4 - x^2 - y^2)^{1/2}\,dx\,dy$$

$$= 8 \int_0^2 \int_0^{\pi/2} (4 - r^2)^{1/2}\, r\, dr\, d\theta$$

$$= 8 \int_0^2 r(4 - r^2)^{1/2}\,dr \cdot \int_0^{\pi/2} d\theta = 8 \cdot I_1 \cdot I_2$$

$$I_1 = \int_0^2 r(4 - r^2)^{1/2}\,dr = \frac{h}{3}\left[r_0(4 - r_0^2)^{1/2} + 4r_1(4 - r_1^2)^{1/2} + r_2(4 - r_2^2)^{1/2}\right]$$

$$= 2.5457$$

$I_2 = \frac{\pi}{4\times 3}\left[1+4+1\right] = \pi/2$ | $k = \frac{\pi/2 - 0}{2} = \frac{\pi}{4}$

$\therefore V = 8 I_1 I_2 = 8 \times 2.5457 \times \frac{\pi}{2}$

$= \underline{32.003084}$ (which is better than Cartesian method)

Alternatively, we use polar co-ords on tripple int.

$$V = 8 \iiint_D dx\, dy\, dz$$

D: $x \geq 0$, $y \geq 0$, $z \geq 0$ & $x^2+y^2+z^2 = 4$

Let $\therefore x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

$|J| = \ldots = r^2\sin\theta$ | $r: 0 \to 2$, $\theta: 0 \to \pi/2$, $\phi: 0 \to \pi/2$

$$\therefore V = 8 \int_0^2\int_0^{\pi/2}\int_0^{\pi/2} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 8 \cdot \int_0^2 r^2 dr \cdot \int_0^{\pi/2} \sin\theta\, d\theta \cdot \int_0^{\pi/2} d\phi = 8 I_1 I_2 I_3$$

$I_1 = \int_0^2 r^2 dr = \ldots = 2.66667$ | $h = \frac{2-0}{4} = 1/2$

$I_2 = \ldots = 1.000134585$ | $h = \frac{\pi/2 - 0}{4} = \pi/8$

$I_3 = \pi/2$

$\therefore V = 8 \times (2.66667) \times (1.0001\ldots) \times \pi/2$

$= 33.5148162$

(which is much more accurate)

Ex Find volume of ellipsoid $\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{4^2} = 1$

Sol: $V = 8 \iiint_D dx\, dy\, dz$

$X = x/2 \qquad Y = y/3 \qquad Z = z/4$

$\Rightarrow x = 2X \qquad y = 3Y \qquad z = 4Z$

$\therefore |J| = \cdots = 24$

$\therefore V = 8 \iiint_{D'} \cancel{8} \cdot dx\, dy\, dz$

$D' : \cancel{X} = 0, \quad Y = 0, \quad Z = 0, \quad X^2 + Y^2 + Z^2 = 1$

Now $X = r \sin\theta \cos\phi \qquad Y = r \sin\theta \sin\phi \qquad Z = r \cos\theta$

$|J'| = r^2 \sin\theta$

$\therefore V = 8 \times 24 \times \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\theta \cdot dr\, d\theta\, d\phi \cdots$

Q Using double integration find volume enclosed by x-y plane, paraboloid $2z = x^2 + y^2$ & cylinder $x^2 + y^2 = 4$

Sol: $V = \iint_D z'\, dx\, dy \quad = \iint \left(\frac{x^2+y^2}{2}\right) dx\, dy$

$x^2 + y^2 = 4 \quad \& \quad z = 0$

$= 2 \int_0^2 \int_0^{\pi} \frac{r^2}{2} \cdot r\, dr\, d\theta$

$x = r\cos\theta$

$y = r\sin\theta$

$|J| = r$

Find the volume bounded by cylinder $x^2+y^2=4$, plane $y+z=4$ & $z=0$

Soln:

$$V = \iint z \, dx\, dy \qquad x^2+y^2=4$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dy\, dx \qquad (y \text{ is odd fun})$$

$$= 2\int_0^2 2\int_0^{\sqrt{4-x^2}} 4\, dy\, dx = 16 \int_0^2\int_0^{\sqrt{4-x^2}} dy\, dx$$

Q Using S/I find mass of solid in the form of +ve octant of sphere $x^2+y^2+z^2=4$ if $f = 2xyz$ at point (x,y,z)

Soln:

$$M = \iiint_D f \, dx\, dy\, dz \qquad D: x \geq 0,\ y \geq 0,\ z \geq 0 \ \& \ x^2+y^2+z^2=4$$

Transform to polar,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$|J| = r^2\sin\theta \qquad r: 0 \to 3 \qquad \theta: 0 \to \pi/2 \qquad \phi: 0 \to \pi/2$

$$\therefore M = \int_0^3\int_0^{\pi/2}\int_0^{\pi/2} 2(r\sin\theta\cos\phi)(r\sin\theta\sin\phi)(r\cos\theta)\cdot r^2\sin\theta \cdot dr\, d\phi\, d\theta$$

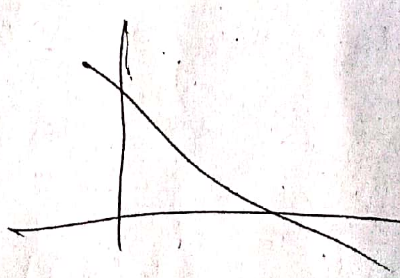
$$= 2 \cdot \int_0^3 r^5\, dr \cdot \int_0^{\pi/2} \sin^3\theta\cos\theta \cdot d\theta \int_0^{\pi/2} \sin\phi\cdot\cos\phi\, d\phi$$

Q. Using $\int^{1/3}$ find $\bar{x}$ where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of region R bounded by the parabolic cylinder $z = 4 - x^2$ & the planes $x=0$, $y=0$, $y=6$ & $z=0$, (Given density is constant).

Sol. $$\bar{x} = \frac{\iiint_D \rho \cdot x \, dx \, dy \, dz}{\iiint_D \rho \cdot dx \, dy \, dz}$$ $z : 0$ to $4-x^2$

$$= \frac{\int_0^2 \int_0^6 \int_0^{4-x^2} x \, dx \, dy \, dz}{\int_0^2 \int_0^6 \int_0^{4-x^2} dx \, dy \, dz}$$

Q. Find the moment of inertia abt x-axis of area enclosed by the lines, (using $\int^{1/3}$) $x=0$, $y=0$ & $\frac{x}{2} + \frac{y}{3} = 1$ Given $\rho = 2xy$



Sol. $$I_x = \iint_R \rho y^2 \, dx \, dy$$

$$= \int_0^2 \int_0^{3(1-x/2)} 2xy \cdot y^2 \, dx \, dy$$

Q. Using 8/15 find M.o.I of uniform spherical ball of mass 'M' & radius 4, about a diameter. (say x - axis)

Soln.
$$I_x = 8 \iiint_D \rho (y^2+z^2) \, dx \, dy \, dz$$

D: $x \geq 0$, $y \geq 0$, $z \geq 0$ & $x^2+y^2+z^2 = 4^2$

$x = r\sin\theta \cos\phi$, $y = r\sin\theta \sin\phi$, $z = r\cos\theta$.

$$\therefore I_x = 8\rho \int_0^4 \int_0^{\pi/2} \int_0^{\pi/2} (r^2 \sin^2\theta \sin^2\phi + r^2\cos^2\theta) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 8\rho \int_0^4 r^4 \, dr \cdot \int_0^{\pi/2}\int_0^{\pi/2} (\sin^3\theta \sin^2\phi + \cos^2\theta \sin\theta) \, d\phi \, d\theta$$

Numerical Solution of Integral Equations:

An eqn. in which the unknown fnc appears under integral sign is known as an integral eqn.

When the limits of integral are const. then we have a Fredholm Integral Equation

(i) $\int_a^b K(x,t) \cdot f(t) \, dt = \phi(x)$ → Fredholm eqn of 1st kind

(ii) $\lambda \cdot \int_a^b K(x,t) \cdot f(t) \, dt = f(x) + \phi(x)$ → 2nd kind

Here, f is the unknown function

Here $f(x)$ occurs in 1st degree i.e, they are linear Fredholm eqn.,

$\phi(x)$ is a known fnx &, so is the kernel $K(x,t)$

If the const. 'b' in (i) & (ii) is replaced by 'x', then ~~the~~ the ~~inte~~ eqns are called Volterra Integral Equations.

(iii) $\int_a^x K(x,t) \cdot f(t)\, dt = \phi(x) \longrightarrow$ 1st kind

(iv) $\lambda \cdot \int_a^x K(x,t) \cdot f(t) \cdot dt = f(x) + \phi(x) \longrightarrow$ 2nd kind.

when, $\phi(x) = 0$ in (ii) & (iv) then they are called non-homogeneous eqns.

For non-homog. eqn ~~the~~ 'λ' is a ~~numerical~~ numerical parameter whereas for homog. eqns 'λ' is an eigen-value parameter. because, in such a case the eqn. represents an eigenvalue problem in which the objective is to find those values of 'λ', called eigenvalues, for which the integral eqn has non-trivial (non-zero) solns called eigen-functions or eigen-vectors.

If the kernel $K(x,t)$ is bounded & continuous then the integral eqn is said to be non-singular.

If the range of integration is infinite or the kernel does not follow the above conditions (bounded & continuous) then the eqns are said to be singular.

To solve the integral eqn. of any kind is to find the unknown function $f(x)$ satisfying the eqn [(i)/(ii)/(iii)/(iv)].

In most of the cases, the solution to an integral eqn. by analytical (mathematical) method is out of the question.

A straightforward numerical approach is to replace the integral eqn. by a system of linear algebraic eqns. (given that the integral eqn is linear in 'f').

We can solve this system by any numerical method for solving a system of linear eqns.

Finite Difference Method :

$$f(x) - \int_a^b K(x,t) \cdot f(t)\, dt = \phi(x) \quad \text{--- (1)}$$

Since a definite integral can be closely approximated by a quadrature formula we approximate the integral in (1) by the formula

$$\int_a^b f(x)\, dx = \sum_{m=1}^{n} A_m \cdot F(x_m) \quad \text{---- (2)}$$

where, A_m & x_m are weights & abscissii respectively,

Now, eqn (1) with the help of (2) reduces to —

$$f(x) - \sum_{m=1}^{n} K(x, t_m) \cdot f(t_m) = \phi(x) \quad \cdots (3)$$

where, $t_1, t_2, \ldots, t_m$ are the points at which the interval $[a, b]$ is subdivided.

Further, eqn (3) must hold for all values of x in the interval $[a, b]$. In particular it must hold for $x = t_1, t_2, \ldots, t_m$;

Thus we have,

$$f(t_i) - \sum_{m=1}^{n} K(t_i, t_m) \cdot f(t_m) = \phi(t_i) \quad \text{--- (4)}$$

$$\text{or, } f_i - \sum_{m=1}^{n} K(t_i, t_m) \cdot f_i = \phi_i \quad \text{--- (4')}$$

Eqn. (4') represents system of n linear eqns which can solved by any known methods.

when the values of $f(t_i)$ are obtained then substituting in (3) we can find approximate ~~value~~ expression $f(x)$ which is the soln. of the integral eqn.

To approximate integrals in the integral eqn any quadrature formula can be applied.

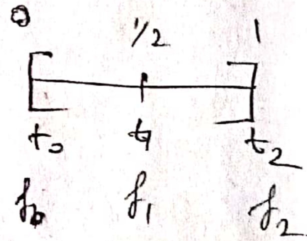
Q Using finite diff method solve

$$f(x) - \int_0^1 (x+t)\, f(t)\, dt = \frac{3}{2}x - \frac{5}{6} \quad \text{---} (1)$$

To approximate integral use i) trapezoidal rule & ii) simpson's 1/3 rd rule

Soln: i) Trapezoidal rule

$[0, 1]$ $\quad h = \frac{1-0}{2} = 1/2$



$f_i = f(t_i)$

$$\therefore f(x) - \frac{h}{2}\Big[(x+t_0) f_0 + 2(x+t_1) f_1 + (x+t_2) f_2\Big] = \frac{3}{2}x - 5/6$$

$$\Rightarrow f(x) - 1/4\Big[(x+0) f_0 + 2(x+1/2) f_1 + (x+1) f_2\Big] = 3/2\, x - 5/6 \quad \text{---} (2)$$

Eqn (2) must hold for $x = t_0, t_1, t_2$ in $x \in [0, 1]$

$\therefore f(t_i) - \frac{1}{4}\left[t_i \cdot f_0 + 2(t_i + \frac{1}{2})f_1 + (t_i+1)f_2\right] = \frac{3}{2}t_i - \frac{5}{6}$

for $\underline{i=0}$ we have, $\underline{i = 0, 1, 2}$

$f_0 - \frac{1}{4}\left[0 \cdot f_0 + 2(0+\frac{1}{2})f_1 + (0+1)f_2\right] = 0 - \frac{5}{6}$

$\Rightarrow 12 f_0 - 3f_1 - 3f_2 = -10$ ---- (4)

for $\underline{i=1}$ we have,

$f_1 - \frac{1}{4}\left[\frac{1}{2} \cdot f_0 + 2(\frac{1}{2}+\frac{1}{2})f_1 + (\frac{1}{2}+1)f_2\right] = \frac{3}{2}\times\frac{1}{2} - \frac{5}{6}$

$\Rightarrow -3f_0 + 12f_1 - 9f_2 = -2$ ---- (5)

for $\underline{i=2}$ we have,

$f_2 - \frac{1}{4}\left[f_0 + 2(1+\frac{1}{2})f_1 + (1+1)f_2\right] = \frac{3}{2} - \frac{5}{6}$

$\Rightarrow -3f_0 - 9f_1 + 6f_2 = 8$ ---- (6)

Solving (4), (5) & (6), $Af = B$

$$[A:B] = \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 12 & -3 & -3 & -10 \\ -3 & 12 & -9 & -2 \\ -3 & -9 & 6 & 8 \end{array}\right]$$

$f_0 = -\frac{7}{6}$, $f_1 = -\frac{5}{6}$, $f_2 = -\frac{1}{2}$

Substituting in eqn (2) we get

$f(x) - \frac{1}{4}\left[x \times (-\frac{7}{6}) + 2(x + \frac{1}{2})(-\frac{5}{6}) + (x+1)(-\frac{1}{2})\right] = \frac{3}{2}x - 1$

$\therefore f(x) = \frac{1}{6}(4x - 7)$

Actual answer (analytical) $\rightarrow$ $\underline{f(x) = x - 1}$

ii) $S\frac{1}{3}$ rule

$$h = \frac{1-0}{2} = \frac{1}{2}$$

$0 \quad 0.5 \quad 1$

$t_0 \quad t_1 \quad t_2$

$$f(x) - \int_0^1 (x+t)\, f(t)\, dt = \frac{3}{2}x - \frac{5}{6}$$

$$\Rightarrow f(x) - \frac{h}{3}\left[(x+t_0)f_0 + 4(x+t_1)f_1 + (x+t_2)f_2\right] = \frac{3}{2}x - \frac{5}{6}$$

$$\Rightarrow f(x) - \frac{1}{6}\left[x\cdot f_0 + 4\left(x+\tfrac{1}{2}\right)f_1 + (x+1)f_2\right] = \frac{3}{2}x - \frac{5}{6}$$

$x = t_0, t_1, t_2$

$$\Rightarrow f_i - \frac{1}{6}\left[t_i\cdot f_0 + 4\left(t_i+\tfrac{1}{2}\right)f_1 + (t_i+1)f_2\right] = \frac{3}{2}t_i - \frac{5}{6} \quad \text{--- (3)}$$

$i = 0, 1, 2$

$i=0$
$\Rightarrow t_i = 0 \quad | \quad 6f_0 - 2f_1 - f_2 = -5$ ---- (4)

$i=1$
$\Rightarrow t_i = 0.5 \quad | \quad -f_0 + 4f_1 - 3f_2 = -1$ ---- (5)

$i=2$
$\Rightarrow t_i = 1 \quad | \quad -f_0 - 6f_1 + 4f_2 = 4$ ---- (6)

Solving (4), (5) & (6) we get,

$f_0 = -1$, $f_1 = -\frac{1}{2}$, $f_2 = 0$

Substituting in eqn (2) we get,

$$f(x) - \frac{1}{6}\left[x\cdot(-1) + 4\left(x+\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) + 0\right] = \frac{3}{2}x - \frac{5}{6}$$

$$\Rightarrow \;\cdots\; \Rightarrow f(x) = x - 1$$

Hence, $S\frac{1}{3}$ gives the actual solution.

Q. Using finite diff method with S 1/3 solve

$$y(x) + \int_0^1 x\,e^{xt}\,y(t)\,dt = e^x \quad \text{--- (1)}$$

Soln. $y(x) + \frac{h}{3}\left[x e^{xt_0} y_0 + 4x e^{xt_1} y_1 + x e^{xt_2} y_2\right] = e^x$

$h = \frac{1-0}{2} = 1/2$

$t_0 = 0$, $t_1 = 1/2$, $t_2 = 1$

$\Rightarrow y(x) + \frac{x}{6}\left[y_0 + 4e^{x/2} y_1 + e^x y_2\right] = e^x \quad \text{--- (2)}$

$x = t_0, t_1, t_2$

$\Rightarrow y_i + \frac{t_i}{6}\left[y_0 + 4e^{t_i/2} y_1 + e^{t_i} y_2\right] = e^{t_i} \quad i = 0, 1, 2$

$i=0$, $\Rightarrow t_i = 0$ | $y_0 + 0 = 1 \Rightarrow y_0 = 1 \quad \text{--- (3)}$

$i=1$, $t_i = 1/2$ | $1.4280\, y_1 + 0.137\, y_2 = 1.5654 \quad \text{--- (4)}$

$i=2$, $t_i = 1$ | $1.0991\, y_1 + 1.4530\, y_2 = 2.5516 \quad \text{--- (5)}$

Solving (3), (4), (5) we get,

$y_0 = 1$, $y_1 = 1.0002$, $y_2 = 0.0995$

Substituting in eqn (2) we have,

$$y(x) = e^x - \frac{x}{6}\left[1 + 4.0008\, e^{x/2} + 0.0995\, e^x\right]$$

which gives the soln. to (1).

Q i) $y(x) - \int_0^1 (x-t)\, y(t)\, dt = \frac{3}{2}x - \frac{7}{6}$

ii) $y(x) + \int_0^1 \frac{y(t)}{1+x^2+t^2}\, dt = 1.5 - x^2$

iii) $y(x) + \int_0^1 x(e^{xt} - 1)\, y(t)\, dt = e^x - x$

iv) $f(x) = \frac{15x - 2}{18} + \frac{1}{3}\int_0^1 (x+t)\, f(t)\, dt$

v) $y(x) = \int_0^1 2x \cdot s \cdot y(s)\, ds - \frac{2x}{3} + 1$

vi) $y(x) + \int_0^{0.5} \frac{(1+s)\cdot y(s)}{2 + \sin \pi(x+s)}\, ds = 1 + \sin \pi x$

vii) $y(x) - \int_0^1 \frac{1+x+s}{2+xs}\, y(s)\, ds = 1 - x^2$

viii) $y(x) - \int_0^1 \frac{1+x+t}{2+x^2+t^2}\, y(t)\, dt = e^{-x}$

Gauss - Legendre 2-points Quadrature formula

$y(x) + \int_0^1 e^{xt} \cdot y(t)\, dt = e^x$ ...... ①

$\int_{-1}^1 f(x)\, dx = \sum_{i=1}^{2} w_i\, f(x_i)$ ----- ②

$= w_1 f(x_1) + w_2 f(x_2)$

$I = \int_0^1 e^{xt} \cdot y(t)\, dt$

$a = 0, \quad b = 1$

$t = \frac{b-a}{2} \cdot u + \frac{b+a}{2}$

$= \frac{1-0}{2} u + \frac{1+0}{2}$

$t = \frac{1}{2}(u+1)$

$dt = \frac{1}{2} du$

$$\therefore I = \frac{1}{2}\int_{-1}^{1} e^{x\left(\frac{u+1}{2}\right)} \cdot y\left(\frac{u+1}{2}\right) \cdot du$$

$t = \frac{u+1}{2}$

$$= \frac{1}{2} \int_{-1}^{1} e^{xt}\, y(t)\, du$$

$a = 0 \quad b = 1$

$t = \frac{1}{2}(u+1)$

$2t - 1 = u$

$t = 0 \quad u = -1$

$t = 1 \quad u = 1$

$$= \frac{1}{2}\sum_{i=1}^{2} w_i \cdot e^{x t_i} \cdot y(t_i)$$

$w_1 = w_2 = 1 \qquad u_2 = -u_1 = 1/\sqrt{3}$

$t_1 = \frac{1}{2}u_1 + \frac{1}{2} = \frac{1}{2}(-1/\sqrt{3}) + \frac{1}{2} = 0.2113249$

$F(u) = c_0 + c_1 u + c_2 u^2 + \dots + c_{2n-1} u^{2n-1}$

$$\int_{-1}^{1} F(u)\, du = [\ ] = 2c_0 + \frac{2}{3}c_2 + \frac{2}{5}c_3 + \dots$$

$F(u_i) = c_0 + c_1 u_i + c_2 u_i^2 + c_3 u_i^3 + \dots + c_{2n-1} u_i^{2n-1}$

$$\int_{-1}^{1} F(u)\, du = \sum_{i=1}^{n} w_i \cdot F(u_i) = w_1 F(u_1) + \dots w_n F(u_n)$$

$$= (w_1 + w_2 + \dots + w_n) c_0 + c_1 (w_1 u_1 + \dots + w_n u_n)$$

$$+ c_2 (w_1 u_1^2 + \dots + w_n u_n^2) + \dots$$

$F(u) = 1, u, u^2 \;\&\; u^3 \qquad w_1 + w_2 = 2$

$w_1 u_1 + w_2 u_2 = 0$

$w_1 u_1^2 + w_2 u_2^2 = 2/3$

$w_1 u_1^3 + w_2 u_2^3 = 0$

$- \quad w_1 = w_2 = 1$

$u_2 = -u_1 = 1/\sqrt{3}$

It can be shown that u_i are the zeroes of the $(n+1)^{th}$ Legendre Polynomial "$P_n(x)$"

we know,

$$(n+1)\, P_{n+1}(u) = (2n+1)\, u \cdot P_n(u) - n P_{n-1}(u)$$

where, $P_0(u) = 1$, $P_1(u) = u$,

$$P_2(u) = \frac{1}{2}(3u^2 - 1)$$

∴ Here $P_2(u) = 0 \Rightarrow \frac{1}{2}(3u^2-1) = 0 \Rightarrow u = \pm$

$P_3(u) = 0 \Rightarrow \frac{1}{2}(5u^3 - 3u) = 0 \Rightarrow u = 0, \pm\sqrt{3/5}$

i) $w_1 = w_2 = 1$ $\qquad w_1 + w_2 = 2$

$w_1 u_1 + w_2 u_2 = 0 \Rightarrow -\frac{1}{\sqrt{3}} w_1 + \frac{1}{\sqrt{3}} w_2 = 0$

$\frac{1}{2}(3u^2 - 1) = 0 \qquad \Rightarrow w_1 = w_2$

$\Rightarrow u = \pm 1/\sqrt{3}$

ii) $u_3 = -u_1 = \sqrt{3/5}$ $\qquad u_2 = 0$

$w_1 = 5/9$, $w_2 = 8/9$, $w_3 = 5/9$

Now, going back to the integral,

$$y(x) + \frac{1}{2}\left[e^{xt_1} y_1 + e^{xt_2} y_2\right] = e^x \quad (2)$$

$t_2 = \frac{1}{2}u_2 + \frac{1}{2}$

$= 0.788675$

$x = t_1$ & t_2

$$\therefore y(t_i) + \frac{1}{2}\left[e^{t_i t_1} y_1 + e^{t_i t_2} y_2\right] = e^{t_i}$$

$i = 1, 2$

$i = 1$

$\Rightarrow t_i = 0.211$ $\quad \Big| \quad y_1 + \frac{1}{2}\left[1.456\, y_1 + 1.813\, y_2\right] = 2.2353$

$i = 2$

$\Rightarrow t_i = 0.78$ $\quad \Big| \quad y_2 + \frac{1}{2}\left[1.1813\, y_1 + 1.8627\, y_2\right] = 2.2005$

Solving system (*) we get, $y_1 = 0.4193$

$y_2 = 1.0104$

Substituting in eqn (2) we get the soln,

$$y(x) = -0.2097 \cdot e^{0.2113x} - 0.5092 \cdot e^{0.7887x} + e^{x}.$$

Q. Solve using 2-point formula.

$$y(x) - \frac{1}{2}\int_0^1 e^{xs} \cdot y(s)\, ds = 1 - \frac{1}{2x}(e^{x}-1)$$

Q. Use 3-point formula to solve

$$y(x) + \int_0^1 e^{xt} \cdot y(t)\, dt = e^{x}$$

Soln: $t = \frac{b-a}{2}u + \frac{b+a}{2} = \frac{1-0}{2}u + \frac{1+0}{2}$

$= \frac{u}{2} + 1/2$ $\therefore dt = \frac{1}{2}du$

$$\therefore I = \frac{1}{2}\int_{-1}^{1} e^{xt} \cdot y(t)\, dt = \frac{1}{2}\sum_{i=1}^{2} W_i \cdot y(t_i)$$

$u_3 = -u_1 = \sqrt{3/5}$, $u_2 = 0$

$W_3 = W_1 = 5/9$ $W_2 = 8/9$

$\therefore t_1 = \frac{1}{2}u_1 + 1/2 = -\frac{\sqrt{3/5}}{2} + 1/2 = 0.1127$

$t_2 = 1/2\, u_2 + 1/2 = 1/2(0) + 1/2 = 0.5$

$t_3 = 1/2\, u_3 + 1/2 = 1/2(\sqrt{3/5}) + 1/2 = 0.8873$

$\therefore y(x) + \frac{1}{2}\left[\frac{5}{4}\cdot e^{x t_1} y_1 + \frac{8}{9}\cdot e^{x t_2} y_2 + \frac{5}{4} e^{x t_3}\cdot y_3\right]$.

Now solving for $x = t_1, t_2, t_3$

$y_1 = 0.4520$, $y_2 = 0.6654$, $y_3 = 1.1546$

Now we can get expression ———

Q Solve given IVP for pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \quad \text{--- } (1)$$

$l = 98$ cm
$g = 980$ cm/s²

$\theta = 0$, $\frac{d\theta}{dt} = 4.472$ at $t = 0$ sec

Find θ , $\frac{d\theta}{dt}$ at $t = 0.2$ using 4th order

Soln: replace $\theta \to y$ & $t \to x$

$\therefore y'' + 10\sin y = 0$; $y(0) = 0$
$y'(0) = 4.472$

Now let $y' = z$; $y(0) = 0$

& $z' = -10\sin y$; $z(0) = y'(0) = 4.472$.

Now solve using 4th RK.

Q In LCR circuit the voltage across capacitor is given by –

$$L.C. \frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = 0$$

Subject to the conditions : $v = v_0$, $\frac{dv}{dt} = 0$ at $t = 0$

Using 4th order RK, compute $v(t)$ when $t = 0.02\,s$; $v_0 = 10\,V$, $C = 0.1\,F$, $L = 0.5\,H$ & $R = 10\,\Omega$

Soln: $LCv'' + RCv' + v = 0$

$\Rightarrow v'' + \frac{R}{L}v' + \frac{1}{LC}v = 0 \Rightarrow v'' + 20v' + 10v = 0$

Now let $v \to y$ & $t \to x$

$\therefore y'' + 20y' + 10y = 0$ $\quad y(0) = 10$, $\quad y'(0) = 0$

$y' = z \Rightarrow z(0) = y'(0) = 0$

Milne's Method :

$$\frac{dy}{dx} = f(x, y, z) \quad \cdots (1)$$

$$\frac{dz}{dx} = g(x, y, z) \quad \cdots (2)$$

x_{n-3}	x_{n-2}	x_{n-1}	x_n	x_{n+1}
y_{n-3}	y_{n-2}	y_{n-1}	y_n	y_{n+1}
z_{n-3}	z_{n-2}	z_{n-1}	z_n	z_{n+1}

Milne's predictor formula is given by, (for y & z)

$$y_{n+1}^{(0)} = y_{n-3} + \frac{4}{3}h\left[2f(x_{n-2}, y_{n-2}, z_{n-2}) - f(x_{n-1}, y_{n-1}, z_{n-1}) + 2f(x_n, y_n, z_n)\right] \quad \cdots (3)$$

$$z_{n+1}^{(0)} = z_{n-3} + \frac{4}{3}h\left[2g(x_{n-2}, y_{n-2}, z_{n-2}) - g(x_{n-1}, y_{n-1}, z_{n-1}) + 2g(x_n, y_n, z_n)\right] \quad \cdots (4)$$

(3) & (4) gives the predictor formula for y & z.

The corrector formulae are

$$y_{n+1}^{(i)} = y_{n-1} + \frac{h}{3}\Big[f(x_{n-1}, y_{n-1}, z_{n-1}) + 4 \cdot f(x_n, y_n, z_n) + f(x_{n+1}, y_{n+1}^{(i-1)}, z_{n+1}^{(i-1)}) \Big] \quad \text{-- (}$$

$$z_{n+1}^{(i)} = z_{n-1} + \frac{h}{3}\Big[g(x_{n-1}, y_{n-1}, z_{n-1}) + 4 \cdot g(x_n, y_n, z_n) + g(x_{n+1}, y_{n+1}^{(i)}, z_{n+1}^{(i-1)}) \Big] \quad \text{-- (}$$

$$i = 1, 2, 3, \ldots$$

We continue until, $y_{n+1}^{(i)} \simeq y_{n+1}^{(i+1)}$ &

$z_{n+1}^{(i)} \simeq z_{n+1}^{(i+1)}$.

Ex: $y' = yz + x \qquad y(0) = 1 \quad \text{------} \; (2)$

$z' = xz + y \qquad z(0) = -1 \quad \text{---} \; (3)$

$x_0 = 0, \quad x_{n+1} = 0.4$ & Given table.

x	0.0	0.1	0.2	0.3
y	1.0	0.9139	0.8522	0.8407
z	-1.0	-0.9092	-0.8341	-0.7705

Sol: $f(x, y, z) = yz + x$; $g(x, y, z) = xz + y$

$h = 0.1 \qquad x_0 = 0, \quad y_0 = 1, \quad z_0 = -1$

x	y	z	f	g
$x_0 = 0.0$	$y_0 = 1.0000$	$z_0 = 1.0000$	$f_0 = -1.0000$	$g_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 0.9139$	$z_1 = 0.9092$	$f_1 = -0.7309$	$g_1 = 0.8230$
$x_2 = 0.2$	$y_2 = 0.8522$	$z_2 = -0.8341$	$f_2 = -0.5108$	$g_2 = 0.6854$
$x_3 = 0.3$	$y_3 = 0.8107$	$z_3 = -0.7705$	$f_3 = -0.3246$	$g_3 = 0.5796$

Now (Predictors)

$$y_4^{(0)} = y_0 + \frac{4}{3}h\left[2f_1 - f_2 + 2f_3\right] = 0.7866$$

$$z_4^{(0)} = z_0 + \frac{4}{3}h\left[2g_1 - g_2 + 2g_3\right] = -0.7174$$

for corrector put $n=3$.

$$y_4^{(1)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + \{y_4^{(0)} z_4^{(0)} + x_4\}\right]$$

$$= 0.7864$$

$$z_4^{(1)} = z_2 + \frac{h}{3}\left[g_2 + 4g_3 + \{x_4 z_4^{(0)} + y_4^{(1)}\}\right]$$

$$= -0.7173$$

$\therefore z_4^{(0)} \simeq z_4^{(1)}$ & ...... $y_4^{(2)} \simeq y_4^{(1)} = 0.7864$

Q. $y'' = (1+x^2)y \longrightarrow \begin{cases} y' = z = f(x,y,z) \\ z' = (1+x^2)y = g(x,y,z) \end{cases}$

Solve for given table

x	0.0	0.1	0.2	0.3
y	1.0	1.0050	1.0202	1.0460
$y' = z$	0.0	0.1005	0.2040	0.3138

Find y_4 & z_4

[Ans: 1.08328, 0.433]

Q $y' = -z$; $z' = y$

$y(0) = 1$, $z(0) = 0$, $x = 0.4\ (0.1)\ 0.6$

x	0.1	0.2	0.3
y	0.9950	0.980025	0.955225
z	0.10000	0.1996	0.296008

Q $x = 1.4\ (0.1)\ 1.6$ $y(1) = 0.77$, $y'(1) = -0.44$

x	y	z
1.1	0.726	−0.473
1.2	0.679	−0.503
1.3	0.629	−0.529

$$y'' + \frac{y'}{x} + y = 0$$

Adams Moulton Method :

$$y' = f(x, y) \quad \text{------} (1)$$

$x_n - x_{n+1}$

$$u = \frac{x - x_n}{h}$$

$$y'(x) = y'\{x_n + (x - x_n)\}$$

$$= y_n' + u\nabla y_n' + \frac{u(u+1)}{2!}\nabla^2 y_n' + \text{-----}$$

$$\int_{x_n}^{x_{n+1}} y' = \int_{x_n}^{x_{n+1}} \{\ldots\} dx$$

$x = x_n + hu$

$dx = h\,du$

$$y_{n+1} - y_n = h\int_0^1 (y_n' + u\nabla y_n' + \text{---})\, du$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{24}\left[2 y_n + 12\nabla y_n' + 10\nabla^2 y_n' + 9\nabla^3 y_n' + \frac{251}{720} \nabla^4 y_n'\right]$$

Truncation error $= O(h^5)$

$$y_{n+1}^{(0)} = y_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right]$$

(The above gives the predictor formula)

$$y_{n+1}^{(i)} = y_n + \frac{h}{24}\left[9 f_{n+1}^{(i-1)} + 19 f_n - 5 f_{n-1} + f_{n-2}\right]$$

where, $f_{n+1}^{(i-1)} = f\left(x_{n+1}, y_{n+1}^{(i-1)}\right)$ — Corrector formula.

Let given system be -

$$y' = f(x, y, z) \quad \& \quad z' = g(x, y, z) \quad \text{--- (1)}$$

The predictor formulae for y & z are -

$$y_{n+1}^{(0)} = y_n + \frac{h}{24}\Big[55 \cdot f(x_n, y_n, z_n) - 59 f(x_{n-1}, y_{n-1}, z_{n-1}) + 37 \cdot f(x_{n-2}, y_{n-2}, z_{n-2}) - 9 f(x_{n-3}, y_{n-3}, z_{n-3})\Big]$$

$$z_{n+1}^{(0)} = z_n + \frac{h}{24}\left[55\, g(x_n, y_n, z_n) - 59 g_{n-1} + 37 g_{n-2} - 9 g_{n-3}\right]$$

The corrector formulae are -

$$y_{n+1}^{(i)} = y_n + \frac{h}{24}\Big[9 f\left(x_{n+1}, y_{n+1}^{(i-1)}, z_{n+1}^{(i-1)}\right) + 19 f(x_n, y_n, z_n) - 5 f(x_{n-1}, y_{n-1}, z_{n-1}) + f(x_{n-2}, y_{n-2}, z_{n-2})\Big]$$

$$z_{n+1}^{(i)} = z_n + \frac{h}{24}\left[9 \cdot g_{n+1}^{(i-1)} + 19 g_n - 5 g_{n-1} + g_{n-2}\right]$$

where, $g_{n+1}^{(i-1)} = g\left(x_{n+1}, y_{n+1}^{(i)}, z_{n+1}^{(i-1)}\right)$

Modified Adams method :

$$y' = f(x,y) \qquad y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + h\, f(x_n, y_n, z_n)$$

$$z_{n+1} = z_n + h\, g(x_n, y_n, z_n) \qquad n = 1,2,3$$

$$y_n^{(0)} = y_n + h f(x_n, y_n)$$

$$y_n^{(i)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n, y_n^{(i-1)})\right]$$

$$y_{n+1}^{(0)} = y_n + h f(x_n, y_n, z_n)$$

$$z_{n+1}^{(0)} = z_n + h\, g(x_n, y_n, z_n)$$

$$y_{n+1}^{(i)} = y_n + \frac{h}{2}\left[f(x_n, y_n, z_n) + f\left(x_{n+1}, y_{n+1}^{(i-1)}, z_{n+1}^{(i-1)}\right)\right]$$

$$z_{n+1}^{(i)} = z_n + \frac{h}{2}\left[g(x_n, y_n, z_n) + g\left(x_{n+1}, y_{n+1}^{(i)}, z_{n+1}^{(i-1)}\right)\right]$$

$$i = 1, 2, 3, \ldots$$

Taylor Series Method :

Ex
$$\left.\begin{aligned} \frac{dx}{dt} &= xy + t \quad ; \; x(0) = 1 \\ \frac{dy}{dt} &= ty + x \quad ; \; y(0) = -1 \end{aligned}\right\} \text{—(1)}$$

$t = 0.1, 0.2, 0.3 \rightarrow$ Compute y. for given t

Sol: $$y(t_0 + (t - t_0)) = y(t_0) + (t - t_0)\, y'(t_0) + \frac{(t - t_0)^2}{2!} y''(t_0) + \ldots$$

$t_0 = 0$

$$y(t) = y_0 + t y_0' + \frac{t^2}{2!} y_0'' + \frac{t^3}{3!} y_0''' + \frac{t^4}{4!} y_0'''' + \cdots$$

$x(t) = x_0 + t x_0' + \frac{t^2}{2!} x_0'' + \cdots$

Now, $x'(t) = x' = xy + t$

$x'(0) = x_0' = x(0) \cdot y(0) + 0 = -1$

$y'(t) = ty + x \Rightarrow y'(0) = y_0' = 0 \times y(0) + x(0) = 1$

$x'' = x'y + xy' + 1$

$\Rightarrow x''(0) = x_0'' = x'(0)\, y_0 + x_0 \cdot y_0' + 1 = 3$

$y'' = y + ty' + x' \Rightarrow y''(0) = \cdots$

Proceeding like this we get $x(t)$ & $y(t)$

$x(t) = 1 - t + \frac{3}{2}t^2 - \frac{7}{6}t^3 + \frac{27}{24}t^4 - \frac{124}{120}t^5 + \cdots$

$y(t) = -1 + t - t^2 + \frac{5}{6}t^3 - \frac{13}{24}t^4 + \frac{47}{120}t^5 \cdots$

$\therefore x(0.1) = 0.9139$ & $y(0.1) = -0.9092$

Q $y'' + xy' + y = 0$ --- (1) $\quad y(0) = 1, \; y'(0) = 0$

$x = 0.1 \,(0.1)\, 0.3$

Sol: $y' = z$ $\Rightarrow z' + xz + y = 0$

(2) $\Rightarrow z' = -(xz + y)$ --- (3)

Now diff n-times using Leibnitz rule,

$y_{n+2} + xy_{n+1} + ny_n + y_n = 0$

$\Rightarrow y_{n+2} + xy_{n+1} + (n+1)\, y_n = 0$

$\underline{x = 0} \Rightarrow (y_{n+2})_0 = -(n+1)\,(y_n)_0$ --- (A)

$n = 0 \Rightarrow (y_2)_0 = -(0+1)\cdot(y_0)_0 = -y_0 = -1$

$n=2 \Rightarrow (y_4)_0 = -(2+1)(y_2)_0 = +3$

$n=4 \Rightarrow (y_6)_0 = -(4+1)(y_4)_0 = -15$

$n=1 \Rightarrow (y_3)_0 = -(1+1)(y_1)_0 = \cdots = 0$

$n=3 \Rightarrow (y_5)_0 = \cdots = 0$

$n=5 \Rightarrow (y_7)_0 = \cdots = 0$

$\therefore \; y(x) = y(0) + x y_0' + \frac{x^2}{2!} y_0'' + \cdots$

$= 1 - \frac{x^2}{2!} + \frac{3}{4!} x^4 - \frac{5\times 3}{6!} x^6 + \cdots$

$= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \cdots$

$\therefore y(0.1) = \cdots = 0.995$

$y(0.2) = 0.9802$ & $y(0.3) = 0.956$

Also $z = y' = -x + \frac{1}{2}x^3 - \frac{1}{8}x^5 + \cdots$

$= -x(\cdots) = -xy$

Now we can use Milne's method to compute $y(0.4)$

Boundary Value Problems :

$y''(x) + p(x) \cdot y'(x) + q(x) \cdot y(x) = r(x)$ —— (*)

$y(a) = \alpha, \; y(b) = \beta$ or $[a, b]$

or, $y'(a) = \alpha, \; y(b) = \beta$

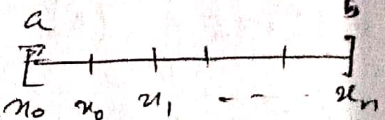
or, $y(a) = \alpha, \; y'(b) = \beta$

or, $y'(a) = \alpha, \; y'(b) = \beta$

or, $c_1 y'(a) \pm c_2 y(a) = \alpha$

$$y''(x) + p(x)\cdot y'(x) + q(x)\cdot y(x) = r(x) \quad \text{---} \ (1)$$

<u>BC</u>: $y(a) = \alpha \quad , \quad y(b) = \beta \quad \text{---} \ (2)$



Let, $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

$x_i = x_0 + ih \qquad i = 0, 1, 2, \ldots, n \qquad \left(h = \frac{b-a}{n}\right)$

Here $x_1, x_2, \ldots, x_{n-1}$ are called mesh points

$y_i = y(x_i) \Rightarrow y_0, y_1, y_2, \ldots, y_n$

Also, $x_{-1} = x_0 - h \ ; \ x_{-2} = x_0 - 2h \ ; \ \ldots$

& $x_{n+1} = x_n + h \ ; \ x_{n+2} = x_n + 2h \ ; \ \ldots$

From (1) we have,

$$y_i'' + p_i \cdot y_i' + q_i \cdot y_i = r_i \quad \text{------} \ (3)$$

$y_i = y(x_i), \quad p_i = p(x_i), \quad q_i = q(x_i), \quad r_i = r(x_i)$

$i = 1, 2, \ldots, n-1$

Here, we do not consider $i = 0$ & $i = n$

Since the value at $i = 0$ & n are known (BC)

[Hence we have $n-1$ unknowns]

$y_{i+1} = y(x_{i+1}) = y(x_i + h)$

$$= y_i + h y_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \frac{h^4}{4!} y_i'''' + \cdots \quad \text{---} \ (4)$$

$$\therefore y_i' = \frac{y_{i+1} - y_i}{h} + O(h) \quad \text{----} \ (5)$$

$y_{i-1} = y(x_{i-1}) = y(x_i - h)$

$$\therefore y_{i-1} = y_i - h y_i' + \frac{h^2}{2!} y_i'' - \frac{h^3}{3!} y_i''' + \frac{h^4}{4!} y_i'''' - + \cdots$$

(6)

$$\therefore \quad y_i' = \frac{y_i - y_{i-1}}{h} + O(h) \quad \text{---- (7)}$$

Subtracting (6) from (4) we get.

$$y_{i+1} - y_{i-1} = 2h\, y_i' + \cdots$$

$$\Rightarrow \quad y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2) \quad \text{.... (8)}$$

Adding (6) & (4) we gets

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + O(h^2) \quad \text{---- (9)}$$

Now

$$y_i''' = \frac{d}{dx}(y'')\Big]_{x=x_i} = \frac{y_{i+1}'' - y_{i-1}''}{2h}$$

$$\therefore \quad y_i''' = \frac{y_{N_1}'' - y_{N_2}''}{2h} \qquad \Bigg| \begin{array}{l} N_1 = i+1 \\ N_2 = i-1 \end{array}$$

$$= \frac{1}{2h}\left[\frac{y_{N_1-1} - 2y_{N_1} + y_{N_1+1}}{h^2} - \frac{y_{N_2-1} - 2y_{N_2} + y_{N_2+1}}{h^2}\right]$$

$$= \frac{1}{2h^3}\left[y_i - 2y_{i+1} + y_{i+2} - (y_{i-2} - 2y_{i-1} + y_i)\right]$$

$$\therefore \quad y_i''' = \frac{1}{2h^3}\left[y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right] \quad \text{--- (10)}$$

Again

$$y_i^{IV} = \frac{d^2(y'')}{dx^2}\Big|_{x=x_i} = \frac{y_{i-1}'' - 2y_i'' + y_{i+1}''}{h^2}$$

$$= \frac{1}{h^2}\left[\frac{y_{N_2-1} - 2y_{N_2} + y_{N_2+1}}{h^2} - 2\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + \frac{y_{N_1-1} - 2y_{N_1} + \cdots}{h^2}\right.$$

$$= \frac{1}{h^4}\left[y_{i-2} - 2y_{i-1} + y_i - 2(y_{i-1} - 2y_i + y_{i+1}) + (y_i - 2y_{i+1} + y_{i+2}) \right]$$

$$\therefore y_i^{IV} = \frac{1}{h^4}\left[y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} \right] \quad \text{---- (11)}$$

Similarly we can proceed for higher orders ...

Now we solve eqn (1),,

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + p_i \frac{(y_{i+1} - y_{i-1})}{2h} + q_i y_i = r_i$$

$$\Rightarrow 2(y_{i-1} - 2y_i + y_{i+1}) + h p_i (y_{i+1} - y_{i-1}) + 2h^2 q_i y_i = 2h^2 r_i$$

$$\Rightarrow (2 - hp_i) y_{i-1} + (-4 + 2h^2 q_i) y_i + (2 + hp_i) y_{i+1} = 2h^2 r_i \quad \text{---- (12)}$$

& $y_0 = \alpha$, $y_n = \beta$ --- (13)

So we have $n-1$ unknowns i.e, $y_1, y_2, \dots, y_{n-1}$

$\underline{\underline{i=1}}$ $(2 - hp_1) y_0 + (-4 + 2h^2 q_1) y_1 + (2 + hp_1) y_2 = 2h^2 r_1$

$$\Rightarrow (-4 + 2h^2 q_1) y_1 + (2 + hp_1) y_2 = 2h^2 r_1 - (2 - hp_1)\alpha$$

$\underline{\underline{i=2}}$

$$(2 - hp_2) y_1 + (-4 + 2h^2 q_2) y_2 + (2 + hp_2) y_3 = 2h^2 r_2$$

$\vdots$

$\underline{\underline{i=n-2}}$ $(2 - hp_{n-2}) y_{n-3} + (-4 + 2h^2 q_{n-2}) y_{n-2} + (2 + hp_{n-2}) y_{n-1}$

$$= 2h^2 r_{n-2}$$

$\underline{\underline{i=n-1}}$

$$(2 - hp_{n-1}) y_{n-2} + (-4 + 2h^2 q_{n-1}) y_{n-1} = 2h^2 r_{n-1} - (2 + hp_{n-1})\beta.$$

Hence the system of $n-1$ simultaneous linear eqn. form a tri-diagonal system -

$$AY = D \quad \cdots \circledast$$

$$A = \begin{bmatrix} A_1 & B_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ C_2 & A_2 & B_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & C_3 & A_3 & B_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C_{n-2} & A_{n-2} & B_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & C_{n-1} & A_{n-1} \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} \qquad D = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{bmatrix}$$

where,

$$A_i = -4 + 2h^2 q_i \qquad i = 1, 2, \ldots, n-1$$

$$B_i = 2 + h p_i \qquad i = 1, 2, \ldots, n-2$$

$$C_i = 2 - h p_i \qquad i = 2, 3, \ldots, n-1$$

$$d_1 = 2h^2 r_1 - (2 - h p_1)\alpha$$

$$d_i = 2h^2 r_i \qquad i = 2, 3, \ldots, n-2$$

$$d_{n-1} = 2h^2 r_{n-1} - (2 + h p_{n-1})\beta$$

Q Solve $y'' + y = 0$ --- (1) given BVP BC

$y(0) = 0, \quad y(1) = 1$ --- (2) in $[0, 1]$

for 4 & 8 subintervals.

Soln: i)

0, 0.25, 0.5, 0.75, 1

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$y_0 = 0, \quad y_4 = 1$

$h = \frac{1-0}{4} = 0.25$

$$\therefore \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i = 0$$

$$\Rightarrow y_{i-1} - (2 - h^2) y_i + y_{i+1} = 0$$

$2 - h^2 = k$

$k = 1.9375$

$$\Rightarrow y_{i-1} - k y_i + y_{i+1} = 0 \text{ --- (3)}$$

$$y_0 = 0, \quad y_4 = 1 \text{ --- (4)}$$

Hence we have the tri-diagonal system

$i=1$ $\quad y_0 - k y_1 + y_2$

$\Rightarrow \quad -k y_1 + y_2 = 0$

$i=2$ $\quad y_1 - k y_2 + y_3 = 0$

$i=3$ $\quad y_2 - k y_3 = -1$

--- (5)

But, $y'' + y = 0 \Rightarrow (D^2 + 1) y = 0$

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$y = A \cos x + B \sin x$ $\qquad y = 0, x = 0$

$\Rightarrow 0 = A \times 1 + 0 \Rightarrow A = 0$

$\Rightarrow y = B \sin x$ $\qquad y = 1, x = 1 \Rightarrow 1 = B \sin 1$

$$\therefore B = \frac{1}{\sin 1} \Rightarrow y = \frac{\sin x}{\sin 1}$$ Exact soln

--- (6)

x	Approx (5)	Exact
0.25	0.2943	0.2940
0.5	0.5701	0.5697
0.75	0.8108	0.8101

obtained from solving (5).

i) Now we take 8 sub intervals. | $h = 0.125$

$$\therefore \; y_{i-1} - k y_i + y_{i+1} = 0 \quad \text{---} \; (8) \qquad i = 1, 2, 3, \ldots, 7$$

$$y_0 = 0 \;, \; y_8 = 1 \quad \text{---} \; (7)$$

Solving system (8) for $i = 1$ to $i = 7$ we ha

x	Approx (8)	Exact (6)
0.125	$y_1 = 0.14817$	0.14816
0.25	$y_2 = 0.29404$	0.29401
	$y_3 = 0.43530$	0.43527
0.5	$y_4 = 0.56976$	0.56974
	$y_5 = 0.69529$	0.69532
0.75	$y_6 = 0.80995$	0.81005
	$y_7 = 0.91297$	0.91214

The results are more accurate than $h = 0.25$ but still can be improved.

Extrapolation to the limit :

$$y_i'' + p_i \cdot y_i' + q_i \cdot y_i = r_i$$

It was found that error T was -

$$T = \frac{h^2}{12}\left[y_i^{IV} + 2p_i\, y_i'''\right] + O(h^4) \Rightarrow \text{proportional to } h^2$$

Richardson's differed approach to the limit —

$$y(x_i) - y_i = h^2 e(x_i) + O(h^4)$$

To extrapolate the limit, solving eqn (1) twice by taking h & $h/2$ respectively —

$$y(x_i) - y_i(h) = h^2 e(x_i) + O(h^4) \quad \text{---}(A)$$

$$y(x_i) - y_i(h/2) = (h/2)^2 e(x_i) + O(h^4) \quad \text{---}(B)$$

$$\Rightarrow 4y(x_i) - 4\, y_i(h/2) = h^2 e(x_i) + O(h^4) \quad \text{---}(B')$$

Subtracting (A) & (B') we gets

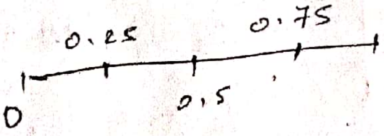
$$y(x_i) = \frac{4\, y_i(h/2) - y_i(h)}{3}$$

Q Solve BVP using finite diff method -

$$y'' + y = 0 \quad \text{---} (1) \qquad y(0) = 0 \;, \; y(\pi/2) = 1 \quad \text{...} (2)$$

Sol: $\because$ π is inconvinient we change interval to $[0, 1]$ & then divide $[0, 1]$ into (i) 4 , (ii) 8 intervals

i)



$$h = \frac{1-0}{4} = 0.25$$

We use transformation - $\quad a = 0 \; ; b = \pi/2$

$$x = (b-a)\,t + a$$

$$= \pi/2 \, t + 0 = \pi/2 \cdot t$$

$$\therefore \quad x \in [0, \pi/2] \Rightarrow t = [0, 1]$$

$$x = \frac{\pi}{2} t \qquad \frac{dx}{dt} = \pi/2$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2}{\pi} \cdot \frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} = \frac{d}{dt}\left(\frac{2}{\pi} \cdot \frac{dy}{dt}\right) \cdot \frac{2}{\pi}$$

$$= \frac{4}{\pi^2} \cdot \frac{d^2y}{dt^2}$$

$$\therefore \quad y'' + \frac{\pi^2}{4} y = 0 \quad \text{--- (3)} \qquad \left| \begin{array}{l} y(0) = 0 \\ y(1) = 1 \end{array} \right\} (4)$$

$$\Rightarrow \quad y_i'' + \frac{\pi^2}{4} y_i = 0$$

$$\Rightarrow \quad \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + \frac{\pi^2}{4} \cdot y_i = 0$$

$$\Rightarrow \quad y_{i-1} - \left(2 - \frac{\pi^2 h^2}{4}\right) y_i + y_{i+1} = 0 \qquad \left| \; 2 - \frac{\pi^2 h^2}{4} = k \right.$$

$$k = 1.8457$$

$$\therefore \quad y_{i-1} - k y_i + y_{i+1} = 0$$

$$i = 1, 2, 3 \qquad \left| \begin{array}{l} y_0 = 0 \\ y_4 = 1 \end{array} \right.$$

$$\left.\begin{array}{rl} -k y_1 + y_2 & = 0 \\ y_1 - k y_2 + y_3 & = 0 \\ y_2 - k y_3 & = -1 \end{array}\right] \Longrightarrow \begin{array}{l} y_1 = 0.3850751 \\ y_2 = 0.7107667 \\ y_3 = 0.9263493 \end{array}$$

Now solving for $H = \frac{1-0}{2}$

(ii)

0.5

$t_0 \quad t_1 \quad t_2$

$\therefore \; y_0 - k y_1 + y_2 = 0$

$y_0 = 0$

$y_2 = 1$

$\Rightarrow -k y_1 = -1$

$\Rightarrow y_1 = \frac{1}{k} = 0.7229875$

$H = 0.5 \qquad H/2 = h = 0.25$

$$\therefore \; y(x_i) = \frac{4\, y_i(H/2) - y_i(H)}{3}$$

$$\Rightarrow y(\;= \frac{4(0.7107667) - (0.7229875)}{3}$$

≈ 0.7

Q $y'' + xy' + y = 3x^2 + 2$ --- (1)

$y(0) = 0 \qquad y(1) = 1$ --- (2)

Soln.

$$\frac{(y_{i-1} - 2y_i + y_{i+1})}{h^2} + x_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + y_i = 3x_i^2 + 2$$

0.25   0.75

0   0.5   1

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$h = \frac{1-0}{4}$

$= 0.25$

$\Rightarrow (2 - hx_i) y_{i-1} + (-4 + 2h^2) y_i + (2 + hx_i) y_{i+1} = 2h^2(3x_i^2 + \ldots)$ ---- (3)

$i = 1, 2, 3$

$i=1$: $-3.875 y_1 + 2.0625 y_2 = 0.27344$ | $y_0 = 0$, $y_4 = 1$

$i=2$: $1.875 y_1 - 3.875 y_2 + 2.125 y_3 = 0.34375$

$i=3$: $1.8125 y_2 - 3.875 y_3 = -1.72656$

Solving above 3 eqns we get,

$y_1 = 0.0624986$; $y_2 = 0.249999$; $y_3 = 0.562498$

Q $y'' = y$ —— (1) $\quad y'(0) = 0, \; y(1) = 1$ $\quad [0, 1]$

Sol: $\dfrac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - y_i = 0$ $\quad h = \dfrac{1-0}{4} = 0.25$

$\Rightarrow y_{i-1} - (2 + h^2) y_i + y_{i+1} = 0$ --- (2)

$i = 0, 1, 2, 3$

$y'_0 = 0, \quad y_4 = 1$ --- (3)

$2 + h^2 = k$

$k = 2.0625$

$$\left.\begin{aligned}
i=0: &\quad y_{-1} - k y_0 + y_1 = 0 \\
i=1: &\quad y_0 - k y_1 + y_2 = 0 \\
i=2: &\quad y_1 - k y_2 + y_3 = 0 \\
i=3: &\quad y_2 - k y_3 = -1
\end{aligned}\right\} \text{---(4)}$$

Also $y'_0 = \dfrac{y_{i+1} - y_{i-1}}{2h}$

Put $i=0$ $\Rightarrow$ $y_0' = \dfrac{y_1 - y_{-1}}{2h} = 0$

$\Rightarrow$ $y_1 = y_{-1}$ --- (5)

Substituting (5) in (4) to eliminate y_{-1},

we get $AY = D$ where,

$$A = \begin{bmatrix} -k & 2 & 0 & 0 \\ 1 & -k & 1 & 0 \\ 0 & 1 & -k & 1 \\ 0 & 0 & 1 & -k \end{bmatrix} \quad Y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$A = LU$ ---- solving system (4),

$y_3 = 0.8396439$, $y_2 = 0.7317655$

$y_1 = 0.6696224$, $y_0 = 0.6493308$

Q. $y'' + y = 0$ $\quad y'(0) = 0$, $y'(\pi/2) = 1$

Soln: $x = (b-a)t + a$ $\qquad h = \dfrac{1-0}{4} = 0.25$

$= \dfrac{\pi}{2} t$ $\Rightarrow$ $t \in [0,1]$ $\qquad t_0 \quad t_4$

$x \in [0, \pi/2]$

$y'(0) = 0$

$\therefore$ $y'' + \dfrac{\pi^2}{4} y = 0$ --- (1) $\qquad y'(1) = \pi/2$ ---

$\therefore$ $y_{i-1} - k y_i + y_{i+1} = 0$ $\qquad i = 0,1,2,3,4$

--- (2) $\qquad k = 1.8457874$

$[y_0' = 0,\ y_4' = \pi/2]$

The system (2) will have two extra unknowns — y_{-1} & y_5 which we have to eliminate.

$y_i' = \dfrac{y_{i+1} - y_{i-1}}{2h} \Rightarrow y_{-1} = y_1$

& $\dfrac{y_5 - y_3}{2h} = y_4' = \pi/2 \Rightarrow y_5 = \text{---}$

Hence the system is $AY = D$

$$A = \begin{bmatrix} -k & 2 & 0 & 0 & 0 \\ 1 & -k & 1 & 0 & 0 \\ 0 & 1 & -k & 1 & 0 \\ 0 & 0 & 1 & -k & 1 \\ 0 & 0 & 0 & 2 & -k \end{bmatrix} \quad Y = \quad D =$$

$y_4 = 0.0104768$, $y_3 = -0.3830299$,

$y_2 = -0.7174688$, $y_1 = -0.9412649$,

$y_0 = -1.0199061$

Q $y'' = y$ $\quad y'(1) = 1.175 \quad y'(3) = 10.018$

$[1, 3] \rightarrow [0, 1]$

Ex: $y'' + y = 0$ — ①

$\left.\begin{array}{l} y'(0) + y(0) = 2 \\ y'(\pi/2) + y(\pi/2) = -1 \end{array}\right\}$ — ②

$[0, \pi/2] \longrightarrow [0, 1]$

$x = 0, \; t = 0$

$x = \pi/2, \; t = 1$

$x = (b-a)t + a$

$= \pi/2 \, t$

0	1/2	1
t_0	t_1	t_2

$y' = \dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{2}{\pi} \cdot \dfrac{dy}{dt}$

& $y'' = \dfrac{d^2y}{dx^2} = \dfrac{4}{\pi^2} \dfrac{d^2y}{dt^2}$

Also, $y'(0) + y(0) = 2 \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} + \left.y\right|_{x=0} = 2$

$\Rightarrow \frac{2}{\pi}\left.\frac{dy}{dt}\right|_{t=0} + \left.y\right|_{t=0} = 2$

$\Rightarrow y'(0) + \frac{\pi}{2} y(0) = \pi$

& $y'(\pi/2) + y(\pi/2) = -1. \Rightarrow \left.\frac{dy}{dx}\right|_{x=\pi/2} + \left.y\right|_{x=\pi/2} = -1$

$\Rightarrow \frac{2}{\pi}\left.\frac{dy}{dt}\right|_{t=1} + \left.y\right|_{t=1} = -1 \Rightarrow y'(1) + \frac{\pi}{2} y(1) = -\pi/2$

Hence, we have the following BVP:

$y'' + \frac{\pi^2}{4} \cdot y = 0$; $y'(0) + \frac{\pi}{2} y(0) = \pi$ --- (4)

$y'(1) + \frac{\pi}{2} y(1) = -\pi/2$ --- (5)

--- (3)

$\therefore \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + \frac{\pi^2}{4} y_i = 0$

$\Rightarrow y_{i-1} - \left(2 - \frac{\pi^2 h^2}{4}\right) y_i + y_{i+1} = 0$

$\Rightarrow y_{i-1} - \left(2 - \frac{\pi^2}{16}\right) y_i + y_{i+1} = 0$. | $K = 2 - \frac{\pi^2}{16}$

$= 1.383165$

$\therefore y_{i-1} - K y_i + y_{i+1} = 0$ $i = 0, 1, 2$ --- (6)

--- (7)

$i=0$: $y_{-1} - k y_0 + y_1 = 0$ --- (8)

$i=1$ $y_0 - k y_1 + y_2 = 0$ --- (9)

$i=2$ $y_1 - k y_2 + y_3 = 0$

| $h = 1/2$

Now $y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$

$\Rightarrow y_0' = \frac{y_1 - y_{-1}}{2h}$ & $y_2' = \frac{y_3 - y_1}{2h}$

Substituting them in B.C. (4) & (5) we get

$y_0' + \frac{\pi}{2} y_0 = \pi$ & $y_2' + \frac{\pi}{2} y_2 = -\pi/2$

$\Rightarrow y_1 - y_{-1} + \frac{\pi}{2} y_0 = \pi$ $\Rightarrow y_3 - y_1 + \frac{\pi}{2} y_2 = -\pi/2$

$\Rightarrow y_{-1} = y_1 + \frac{\pi}{2} y_0 - \pi$ --- (10) $\Rightarrow y_3 = y_1 - \frac{\pi}{2} y_2 - \frac{\pi}{2}$ --- (11)

Using (10) & (11) in system (7, 8, 9) we get,

$y_1 - ky_0 + (y_1 + \frac{\pi}{2} y_0 - \pi) = 0$

$\Rightarrow 2y_1 + (\frac{\pi}{2} - k) y_0 = \pi$ --- (12)

& $-(k + \pi/2) y_2 + 2y_1 = \pi/2$ --- (13)

Solving the system (12), (13) & (8),

$$\begin{bmatrix} \pi/2 - k & 2 & 0 \\ 1 & -k & 1 \\ 0 & 2 & -(k+\pi/2) \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \\ \pi/2 \end{bmatrix}$$

$y_0 = 1.5289$, $y_1 = 1.4264$, $y_2 = 0.4340$.

Q i) $y'' + y = 0$; $y'(0) + y(0) = 1$

$y'(\pi/2) - y(\pi/2) = 0$

$[h = 1/4 = 0.25]$

ii) $\frac{d^2y}{dx^2} + \frac{y}{4} = 0$, ~~y(0)=0~~ $h = 0.25$

a) $y'(0) = 0$, $y'(\pi) = 1$

b) $y'(0) = 0$, $y'(2\pi) = 0$

iii) $y''' - y' = e^x$; $y(0) = 0$, $y(1) = 1$, $y'(1) = 0$.

iv) $y'' - xy' + x^2 y = x^3$; $y(0) + y'(0) + y(1) - y'(1) = 4$

[$h = 0.25$] $\quad y(0) - y'(0) + y(0) - y'(1) = 3$

Higher Order derivatives : [Deflection of Beams]

Q Solve BVP for beam built-in at $x = 0$ & simply supported at $x = 3$. Compute y at pivotal points $x = 1, 2$ by finite diff method. Given length of beam = 3 metres.

$$\frac{d^4 y}{dx^4} + 2y = \frac{1}{9}x^2 + \frac{2}{3}x + 4$$

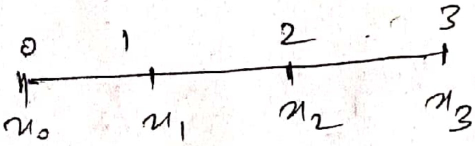
Soln: Clamped at $x = 0$ (left end) [Clamped ⟺ Built-in]

⇒ no displacement & no velocity.

⇒ $y(0) = 0$, $y'(0) = 0$

Simply supported at $x = 3$ ⇒ $y(3) = 0$, $y''(3) = 0$

acceleration is zero



$h = \frac{3-0}{3} = 1$

Note: If right end is free then the BC at $x = 3$ is $y''(3) = y'''(3) = 0$

Q $y^{IV} + 81y = 729x^2$ ... (1) [Clamped at $x=0$, Free at $x=1$]

B.C. : $y(0) = y'(0) = y''(1) = y'''(1) = 0$ ... (2)

Soln: $h = \frac{1-0}{3} = 1/3$.