GRAPH THEORY

Graph: A graph denoted by $G(V, E)$ is a 2-tuple where $V \neq \phi$ & $E \subseteq V \times V$

Pictorial Representation: For each $x \in V$, create a dot & add a curve between two dots x & y if $(x, y) \in E$ where,

V is the set of nodes / vertices

E is the set of edges

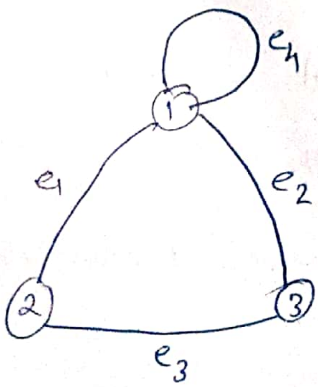
Undirected graph: A graph in which edges are not directed. $(x, y) \in E \Rightarrow (y, x) \in E$

Directed graph: A graph in which the edges are directed. $(x, y) \in E \not\Rightarrow (y, x) \in E$

Simple graph: A graph which is without any loops & multiedges. We generally consider simple, undirected graphs for further purposes

Eg: $V = \{1, 2, 3\}$ $\quad E \subseteq V \times V$

$E = \{(1,2), (1,3), (1,1), (2,3)\}$



Adjacent: Two vertices are said to be adjacent if there exists an edge between them.

Two edges are adjacent if they share a common ~~end~~ end point.

Incident: If $e = (x, y)$ is an edge, then we say that e is incident on x & y

Neighbour of a Vertex: In a graph $G = (V, E)$ a vertex $u \in V$ is said to be neighbour of vertex $v \in V$ if $(u, v) \in E$

~~Neigh~~ Neighbourhood: $N_G(v)$ = set of neighbours of v in graph G.

$$N_G(v) = \{u \in V : (u, v) \in E\}$$

Closed Neighbourhood: $N_G[v] = N_G(v) \cup \{v\}$

Degree of a Vertex: $d_G(v) = |N_G(v)|$

Minimum degree of a graph:

$$\delta(G) = \min.\{d_G(v) : v \in V\}$$

Maximum degree of a graph:

$$\Delta(G) = \max.\{d_G(v) : v \in V\}$$

Corrollary: $\delta(G) \leq Avg_d(G) \leq \Delta(G)$

where, $Avg_d(G)$ is the average degree.

Ex: Does the equality always holds true?

Regular Graph : A graph in which all degrees are equal

Complete graph : A graph in which every pair of vertices are adjacent.

Fundamental Theorem of Graphs –

If graph G is simple then | G(V,E)

$$\sum_{v \in V} d_G(v) = 2|E(G)|$$

Proof: Take any edge xy of G

Let $S = \sum_{v \in V} d_G(v)$ then,

the edge xy contributes a value of 2 to S

$\therefore$ $S = 2|E(G)|$ or $2|E|$ (proved)

Let V_o = set of vertices having odd degree

& V_e = set of vertices having even degree

$$\therefore \; S = \sum_{v \in V_o} d_G(v) + \sum_{v \in V_e} d_G(v) = 2|E|$$

Corr: The no. of vertices have odd degree is even.

or, $|V_o|$ = even,

Complement of a Graph : Given a graph G(V,E), the complement of G, denoted by $\bar{G}$ or G^c is the graph with vertex set $V^c = V$ & edge set $E^c = \{xy \mid xy \notin E\}$

<u>Walk</u>: Any sequence of vertices & edges of the form $v_0 e_1 v_1 e_2 v_2 \ldots v_n$. Simply called as a $u-v$ walk.

<u>Path</u>: A walk in which no vertex is repeated. Also called a $u-v$ path

<u>Trail</u>: No edge is repeated.

<u>Cycle</u>: It is a closed $u-v$ path $(u=v)$

Circuit / <u>Tour</u>: It is a closed $u-v$ trail

<u>Lemma</u>: Every $u-v$ walk contains a $u-v$ path.

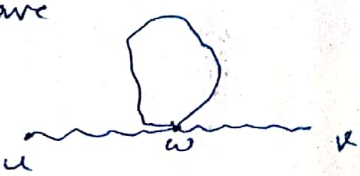
<u>Proof</u> Define length of a walk = no. of edges

For $l=0$ or $l=1$ the lemma is true

Let the lemma be true for $l=k$ where $k>1$.

If the k length walk is a path itself then the lemma is true.

If not then it must have a loop at some vertex 'w'.



So deleting the edges of the loop at w we get a $u-v$ walk with length $< k$ (for which lemma is true)

Hence, by strong induction, lemma is true

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1
2
3
k

Connected : A vertex u is connected to v if $\exists$ a $u-v$ path.

A graph G is said to be connected if for every u & v of G, $\exists$ a $u-v$ path

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Subgraph : For graph $G(V, E)$ a graph $H(V', E')$ is said to be a subgraph of G iff $V' \subseteq V$ & $E' \subseteq E$

Induced Subgraph : For graph $G(V,E)$ & $S \subseteq V$ $G[S]$ is the induced subgraph on G by S given by,

$G[S] = (S, E_S)$ where, $E_S = \{xy \in E \mid x, y \in S\}$

Component / Connected Component : A component of a graph is a maximal connected subgraph of G.

Cut-Vertex : A vertex v in G is said to be a ~~cut~~ cut vertex if the removal of v increases the number of components in $G - v$.

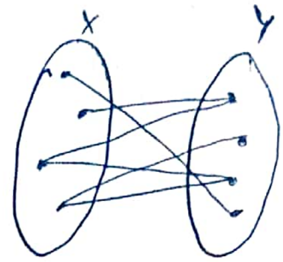
Cut-Edge / Bridge : An edge e in G is said to be a bridge if its removal increases the no. of components in $G - e$

! **Note :** If uv is an edge then "u is connected to v" is a weak statement, it is better to say "u is adjacent to v"

! **Bipartite Graph :** $G(V, E)$ s.t.

$V = X \cup Y$ & $X \cap Y = \phi$

For every edge $e \in E$, one end point of e lies in X while the other lies in Y.



Complete bipartite graph K_{n_1, n_2} is a bipartite graph with $|X| = n_1$, $|Y| = n_2$ having all possible edges.

A bi-partite set may be denoted by $G(X, Y, E)$ where X & Y are the partite sets of G.

K-dimensional Cube / Hypercube of order k.

It can be thought of as a graph Q_k

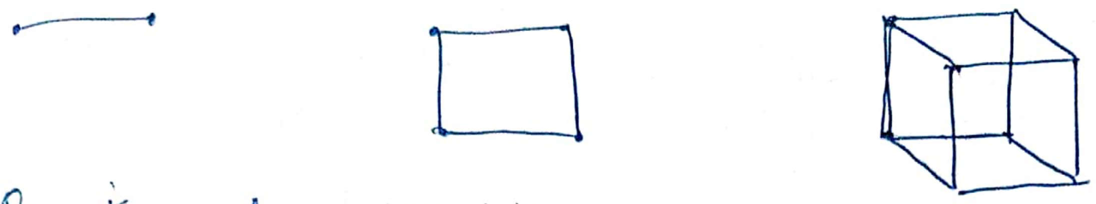
$$V(Q_k) = \{ (a_1, a_2, \ldots, a_k) \mid a_i \in \{0, 1\} \}$$

u, v are adjacent if they differ in exactly one position (hamming distance $= 1$)

$$|V(Q_k)| = 2^k$$

It can be seen that Q_k is a k-regular graph with $d_{Q_k}(v) = k$

$k=1$ $\quad$ $k=2$ $\quad$ $k=3$



Q_k is also bipartite.

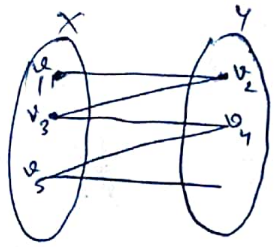
<u>Lemma</u>: A graph G is bipartite iff G contains no odd cycle.

<u>Proof</u>: Let G be a bipartite graph & C be a cycle of odd length in G.

$C = v_1 v_2 \dots v_k v_1$ where, k = odd

without loss of generality,

let $v_1 \in X \Rightarrow v_k \in X$



[$\because$ k is odd]

$\Rightarrow v_1, v_k \in X$ & $v_k \!-\!\!-\! v_1 \in E$ which is a <u>contradiction</u>.

Conversely, let G does not contain odd cycles

let H be a component of G

$\Rightarrow$ H doesn't contain any odd cycle.

let $u \in V(H)$ & $f(v)$ = shortest path from u to v

Define, $X_H = \{ v \in V(H) \mid f(v) \text{ is even} \} \cup \{u\}$

$Y_H = \{ v \in V(H) \mid f(v) \text{ is odd} \}$

Clearly, $X_H \cap Y_H = \phi$

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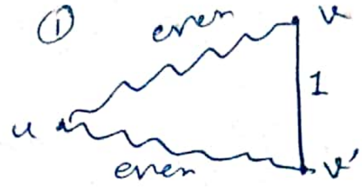
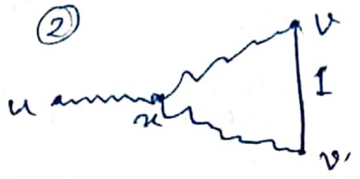
Let $v, v' \in X_H$ s.t. $vv' \in E(H)$

then, we will get a closed odd walk

i.e., $u \to v$, $v - v'$, $v' \to u$



Since every closed walk contains a closed path we get a cycle in H



Case - 1 : If the $u \to v$ walk ~~path~~ does not intersect $v' \to u$ walk

then the length of cycle is even + even + 1

= odd which is a contradiction.

Case - 2 : If the intersection is non - empty

Let x be the vertex in $W_{u,v} \cap W_{u,v'}$

s.t. $W_{x,v} \cap W_{x,v'} = \phi$ then the length

of cycle = even $- |W_{u,x}|$ + even $- |W_{u,x}| + 1$

= even $- 2k + 1$ = even + 1 = odd

[again a contradiction].

Hence there cannot be any edge

$vv' \in E(H)$ s.t. $v, v' \in X_H$.

Similarly there is no edge in Y_H

Hence the component H is bipartite. &

similarly other components are also bipartite.

A sequence of n numbers is said to be a ~~degree sequence~~ graphic if $\exists$ a graph having degree sequence as that sequence d

where $d : d_1 \geq d_2 \geq d_3 \geq \cdots \geq d_n$

Eg $\cancel{5}, 5, 4, 3, 2, 2, 2, 1$ (Delete vertex with degree 5 & subtract 1 from next 5 vertices)

$\cancel{4}, 3, 2, 1, 1, 2, 1$

$2, 1, 0, 0, 2, 1$

$= \cancel{2}, 2, 1, 1$

$= 0, 1, 0, 1 = 1, 1 \Rightarrow$ 

Hence d is a graphic

Havel - Hakimi Theorem:

$d: d_1 \geq d_2 \geq \cdots \geq d_n$ is a graphic

$\Leftrightarrow$ d' is graphic where d' is obtained from d by removing d_1 & subtracting 1 from the next d_1 numbers.

Proof: $(\Leftarrow:)$ Given d' & G' (one graph having d' as degree seq.), we introduce a new vertex 'x' s.t. 'x' is adjacent to the vertices with degrees $d_2-1, \ldots, d_{d_1+1}-1$. Then $\exists$ a graph with dep. seq. 'd'.

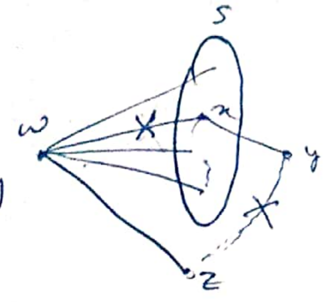
$(:\Rightarrow)$ Let G be a graph with dep. seq 'd'

Let S = the set of vertices with degrees $d_2, d_3, \ldots, d_{d_1+1}$ in G

Let 'w' be the vertex in G with degree d_1 i.e., $d_G(w) = d_1$

If the neighbours of w form the set S then the proof is completed by deleting w.

$N_G(w) = S \Rightarrow$ trivial



Let G be chosen such that $|N_G(w) \cap S|$ is maximized (Assumption)

Claim: $N_G(w) = S$

Proof: If $N_G(w) \neq S$ then $\exists\ x \in S$ & $z \notin S$ st. $wz \in E$, $wx \notin E$.

Also, $d_G(x) > d_G(z)$ (else z must replace x in S)

Let 'y' be a vertex st. $xy \in E$, $yz \notin E$.

Such a y exists because $d_G(x) > d_G(z)$

Construct $G_1 = (G - \{wz, xy\}) \cup \{wx, yz\}$

The degree seq. of G_1 is same as G &

moreover, $|N_{G_1}(w) \cap S| > |N_G(w) \cap S|$

which is a contradiction.

$\therefore N_G(w) = S$

$\therefore G' = G - w$ & G' has degree seq. d'

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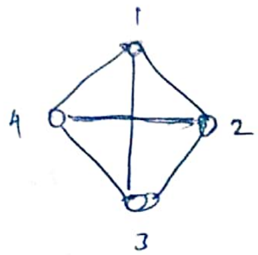
Ex. If $0 \leq d_G(v) \leq n-1$ (n vertices) Is it possible to have graph with distinct degrees. If not which vertex has repeated degrees?

Adjacency Matrix: For a graph $G(V,E)$ the adjacency matrix $A(G)$ is given by-

$$A(G)_{ij} = \begin{cases} 1 & ; \text{ if } ij \in E \\ 0 & ; \text{ otherwise} \end{cases}$$

Eg.



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

For simple graphs, $A(G)$ is a symmetric matrix with diagonal entries $= 0$.

$$\sum_{j=1}^{n} a_{ij} = d_G(v_i)$$

Degree Matrix: $D = \begin{bmatrix} d_1 & & 0 \\ & d_2 \ddots & \\ 0 & & d_n \end{bmatrix}$

Laplacian Matrix: $L = D - A$

Hence, L is also symmetric matrix. Let the eigen values of L be $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. If $\lambda_2 > 0$ then the graph G is connected.

Incidence Matrix : $M(G)_{ij} = \begin{cases} 1 ; & \text{if } v_i \text{ is endpt. of } e_j \\ 0 ; & \text{otherwise} \end{cases}$

It is an $n \times m$ matrix, where m is the no. of edges.

Note : In A^k the ij element gives the no. of walks of length k from ij

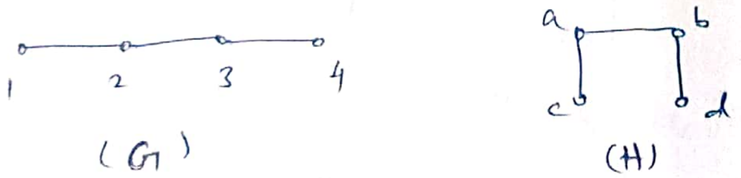
Girth : Length of the ~~longest~~ shortest cycle in graph.

Isomorphism : Two graphs G, H having function $f : V(G) \longrightarrow V(H)$ is said to be an isomorphism if $\forall$ $x, y \in V(G)$

$$xy \in E(G) \iff f(x) f(y) \in E(H)$$

& G, H are called isomorphic graphs. They are denoted by $G \cong H$.

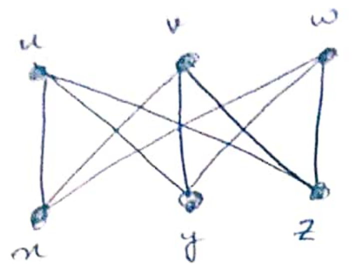
Eg



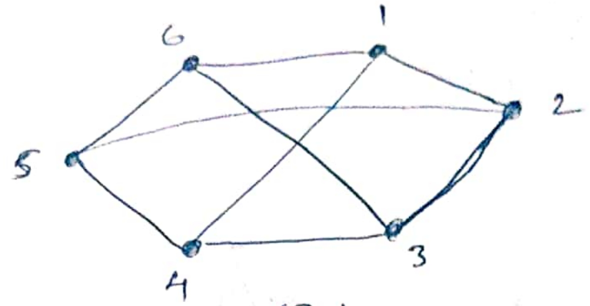
$f(1) = c$; $f(2) = a$; $f(3) = b$; $f(4) = d$ $\therefore G \cong H$

Note : $G \cong H \iff G^c \cong H^c$

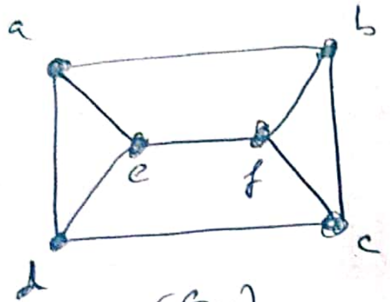
Ex:



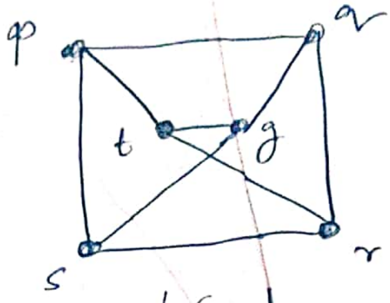
(G_1)



(G_2)



(G_3)



(G_4)

Here, $G_1 \cong G_2 \cong G_4$; $G_3 \not\cong G_4$

Self - Complementary Graph : $G \cong G^c$

Eg: P_4 —  C_5 - 

Let $n = |V(G)|$ in $G(V, E)$

$G \cong G^c \Rightarrow |E(G)| = |E(G^c)|$

$\Rightarrow 2|E(G)| = \dfrac{n(n-1)}{2}$ $\left[\because G \cup G^c \text{ is } K_n\right]$

$\Rightarrow |E(G)| = \dfrac{n(n-1)}{4}$

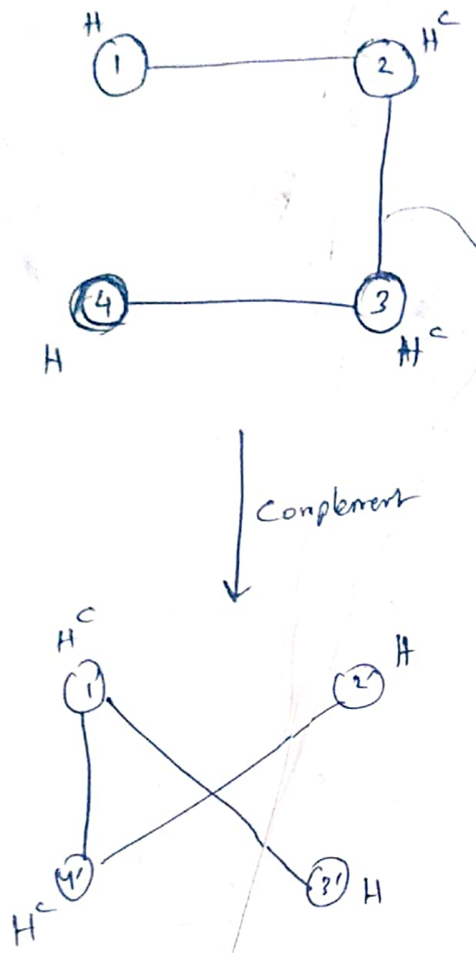
$\Rightarrow n \equiv 0, 1 \pmod 4$

$\therefore G \cong G^c \Rightarrow n \equiv 0, 1 \pmod 4$

Conversely,

Let $n \equiv 0 \pmod 4$ $\Rightarrow$ $n = 4k$, $k = 1, 2, \ldots$

$\therefore$ Construct graph G s.t.



We group k vertices together & form 4 such groups.

Edge represents all possible edges crossing subgraph ② & ③.

Here H, H^c are subgraphs inside the k-sized groups.

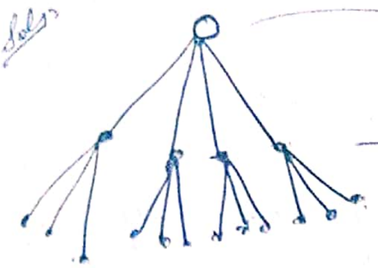
From P_4 analogy we can see it is self complement.

Similarly, for $n \equiv 1 \pmod 4$ we use C_5 analogy.

$\therefore$ If $n \equiv 0, 1 \pmod 4$ then $\exists$ a self-complementary graph with n vertices.

Q Let G has a girth of 5. If $\delta(G) = k$, then G must have at least k^2+1 vertices.

Sol:



→ Let this be any vertex

→ It must have atleast k neighbours (none of which can be adjacent) $5(k-1)$

→ Each can again have k or more neighbours which may be adjacent. (giving a cycle of 5)

minimum no. of vertices $= k^2+1$.

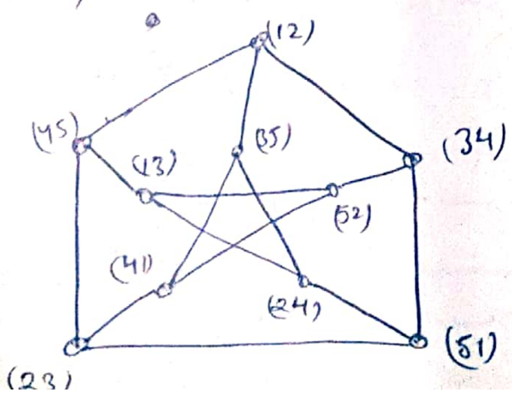
Petersen Graph: $S = \{1, 2, 3, 4, 5\}$

S_2 = set of two-element subsets of S ~~such that no two of them have common element~~

$= \{\{1,2\}, \{1,3\}, \ldots\}$

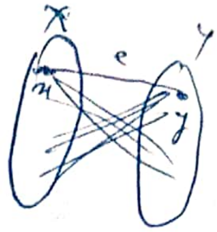
$\boxed{K_n}$

$\exists$ edge between two vertices $\{a,b\}$ & $\{c,d\}$ if their intersection is empty.



T-1

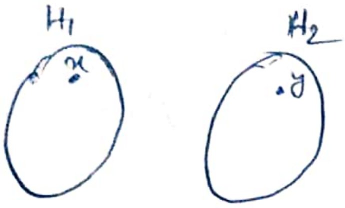
15. G is regular bipartite graph $k \geq 2$



Let G have a cut edge 'e'

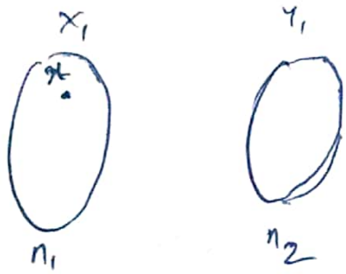
$\Rightarrow G - \{e\}$ is disconnected.

Let H_1 & H_2 be the components in $G - \{e\}$



Now, deleting an edge cannot create an odd cycle so H_1 & H_2 must also be bipartite.

Consider H_1,



In H_1 x is connected to $k-1$ vertices from Y; other elements in X_1 are having degree k (unchanged)

$$\therefore \; k-1 + (n_1 - 1)\cdot k = k n_2$$

$$\Rightarrow k(n_1 - n_2) = 1 \quad (\Rightarrow\Leftarrow)$$

Hence, G has no cut edge.

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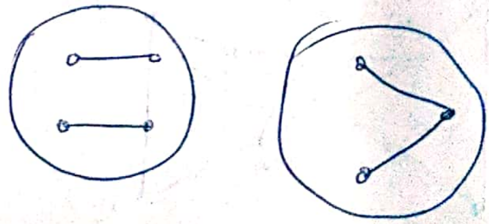
9. G does not have isolated node

An induced subgraph with exactly two edges $\Rightarrow$ there is one of following two structures in graph:



Now let G be not complete

$\Rightarrow \exists\ x, y \in V(G)$ such that $xy \notin E(G)$

Case - 1 x & y have a common neighbour

Then the induced subgraph will have two edges

$(\Rightarrow \Leftarrow)$.

Case - 2 : x & y do not have common neighbour.

Let a & b are neighbours of x & y resp.

(Note that there cannot be an isolated vertex)

Now there must be an edge btw a & b otherwise it will give an induced ~~vertex~~ subgr of two edges

But, (x, a, b) will form ( ) induced subgr $\Rightarrow$ x & b must be connected

which means $\Rightarrow \Leftarrow$.

Hence G must be complete.

Cut : Set of edges whose deletion increases the no. of components in a graph

Cut - set / Separating set / separation : Set of vertices whose deletion increases no. of components in graph

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Lemma:

An edge 'e' is a cut-edge iff 'e' is not contained in any cycle.

Proof ($\Leftarrow$:) Let e belong to component H of graph G. Deletion of edge 'e' has no effect on other components.

Let $e = xy$ is not contained in any cycle

$\Rightarrow$ $\exists$ a unique path from x to y in H which is the edge xy

$\Rightarrow$ $H - \{e\}$ is disconnected

$\Rightarrow$ e is a cut-edge.

($\Rightarrow$:) e is a cut-edge

$\Rightarrow$ $H - \{e\}$ is disconnected & $\nexists$ any path

If 'e' were to be in a cycle then there must be a path between x and y in $H - \{e\}$. But there isn't.

Hence, e is not a part of any cycle in H.

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TREES :

Acyclic graph : The graph without cycles

Tree : Connected & acyclic graph

Forest : It is an acyclic graph.

A vertex of degree one in a tree is called a leaf.

Lemma : Every tree T has atleast 2 leaves.

Proof : Proof by induction on n = no. of vertices in tree

$n=2$  Only one tree possible. It has 2 leaves.

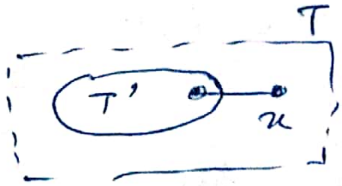
$n=3$  Only one tree possible. It has two leaves.

Assume, $n=k$ & tree with k vertices has atleast two leaves.

$n=k+1$ Let T be a tree with $k+1$ vertices

[There must be atleast one leaf. otherwise $\delta(T) \geq 2 \Rightarrow \exists$ cycle in T].

Let u be a leaf in T.



Let $T' = T - \{u\}$

$\therefore$ T' is connected & acyclic $\Rightarrow$ T' is tree with k vertices

By induction hypothesis,

T' contains leaves 'u' & 'v'

Now, if 'x' is not adjacent of 'u' & then 'u' & 'v' are also leaves of T

$\therefore$ x has degree 1 in T $\Rightarrow$ x can be adjacent to both u & v

if x is adjacent to u then x & v are leaves of T, & if x is adjacent to v then x & u are leaves of T

Hence by induction, T must have atleast two ~~edges~~ leaves.

Lemma: Every tree T with n vertices has (n-1) edges.

Proof: Proceeding as previous proof

Let lemma be true for n = k vertices

n = k+1 Let T be tree with k+1 vertices

T has a leaf, say, x

$T' = T - x$

$\Rightarrow$ T' is connected & acyclic

T' has k vertices $\Rightarrow$ it has k-1 edges

Since we removed one edge while removing x (as it has a leaf)

$\therefore$ T has $k-1+1 = k$ edges (proved)

4. (a) The following are equivalent about a graph G.

i) G is a tree (connected & acyclic)

ii) G has $n-1$ edges & is connected

iii) G has $n-1$ edges & is acyclic

iv) $\exists$ a unique path between every pair of vertices in G.

Proof (i) $\Rightarrow$ (ii)

Given 'G' is a tree then it is connected.

Also it has $n-1$ edges ~~&~~ (can be proved using ~~induct~~ induction)

(ii) $\Rightarrow$ (iii)

Given G has $n-1$ edges & it is connected

If G has a cycle then deleting an edge from the cycle still keeps the ~~remaining~~ graph connected but reduces number of edges to $n-2$

~~of graph~~ But a simple graph of n vertices must have atleast $n-1$ ~~no~~ edges to be connected ($\Rightarrow \Leftarrow$)

(iii) $\Rightarrow$ (iv)

Suppose graph having $n-1$ edges & acyclic has k components. $(n = n_1 + n_2 + \cdots + n_k)$

$\because$ Each component must be acyclic

$\Rightarrow$ Each comp must have $n_i - 1$ edges.

$\therefore$ Total no. of edges $= \sum n_i - 1 = n-k$

$\Rightarrow$ $k=1$ (connected) at least one path btw every pair

~~But~~

Now if there is more than one path b/w any pair of vertices then it must have cycle(s)

∴ There is exactly one path b/w every pair

(iv) ⇒ (i)

∃ unique path ⇒ G is connected & acyclic

<u>Spanning Subgraph</u> : A subgraph H of G is said to be a spanning subgraph of G if $V(H) = V(G)$

<u>Spanning Tree</u> : A spanning subgraph that is a tree. G must be connected for spanning tree to exist.

<u>Theorem</u> : Let T & T' be different spanning trees of G & $e \in E(T) - E(T')$ then ∃ $e' \in E(T') - E(T)$ s.t. $(T-e) \cup \{e'\}$ is also a spanning tree of G

<u>Proof</u> : Since every edge of T is a cut edge, $T-e$ has two components say U & U'

Since T' is connected, ∃ an edge $e' = xy$ s.t. $x \in U$ & $y \in U'$

Moreover $e' \notin E(T)$ [∵ T was acyclic]

⇒ $(T-e) \cup \{e'\}$ is a spanning tree of G.

Theorem: If T & T' are diff spanning trees of G & $e \in E(T) - E(T')$ then $\exists$ $e' \in E(T') - E(T)$ s.t. $(T' \cup \{e\}) - \{e'\}$ is also S.T. of G.

Proof: $T' \cup \{e\}$ contains a cycle C [which contains e]

Pick any edge e' from $E(C) - E(T) \neq \phi$

then $e' \in T'$. Removing e' ~~makes~~ breaks the cycle C $\Rightarrow$ $(T' \cup \{e\}) - \{e'\}$ is a ST.

Theorem: If T is a tree with k-edges & G is a simple graph with $\delta(G) \geq k$ then T is a subgraph of G.

Proof By induction on k.

$k=0$ [isolated vertex & $\delta(G) \geq 0$ $\Rightarrow$ pick any vertex with $d_v = 0$]

Let the theorem be true for tree of $k-1$ edges

Now, Consider a tree with k-edges, T

$\exists$ a leaf in T, say 'y' & let 'u' be the neighbour of 'y' in T.

Let $T' = T - \{y\}$

$\therefore |E(T')| = k-1$

$\Rightarrow$ T' is isomorphic to some subgraph in G.

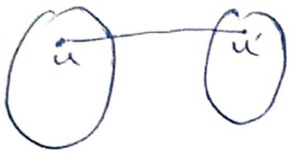
$\delta(G) \geq k > k-1$ $\quad$ $u \in T'$ & $d_u(G) > k-1$

$\Rightarrow$ $\exists$ vertex v in G s.t. $uv \in E(G)$ &

$v \notin T'$ $\Rightarrow$ v is isomorphic to y. (proved)

Ex: Let G be a graph with max degree of G being 'Δ'.
Construct a graph G' from G s.t.
$d(v) = \Delta \quad \forall \; v \in V(G')$

Soln: Let u be any vertex with degree $< \Delta$ then copy the graph and make connections b/w the vertices having degree $< \Delta$



repeat this process ---

Distance: Distance between any two nodes 'u' & 'v' in a graph is the length of the shortest path between u & v.
It is denoted by $d_G(u, v)$.

Diameter: $\operatorname{diam}(G) = \max_{u,v \in V} \{ d_G(u,v) \}$

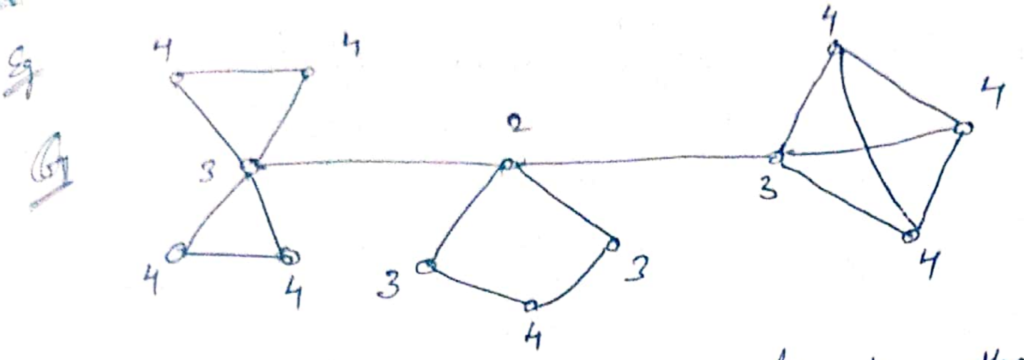
Eccentricity: $\varepsilon(u) = \max_{v \in V} \{ d_G(u,v) \}$

(It is the max dist vertex from a vertex)

Radius: $\operatorname{rad}(G) = \min_{u \in V} \{ \varepsilon(u) \}$

Center: The subgraph induced by the set of vertices having minimum eccentricities.

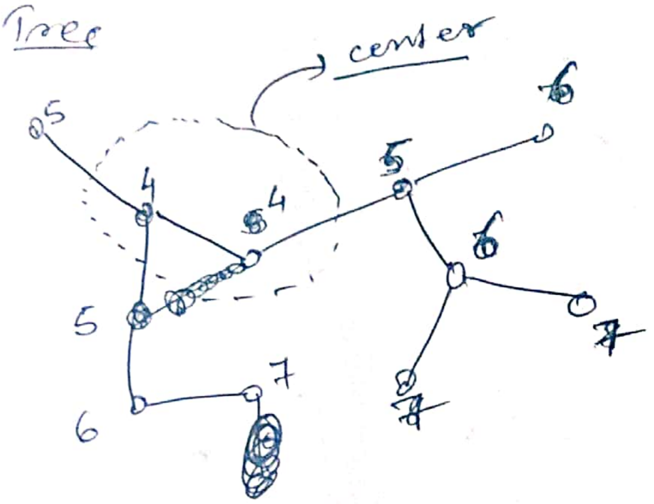
Eg



The nos. on vertices denote their ~~eccent~~ eccentricities.

$\therefore$ $\mathrm{diam}(G) = 4$ $\qquad$ $\mathrm{rad}(G) = 2$

Eg: Tree

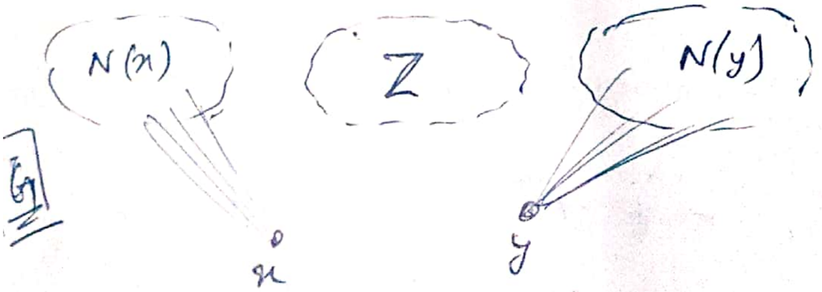


Ex: The center of a tree is a vertex or an edge.

Eg Petersen Graph — diameter = 2

Hypercube Q_k — diameter = k

Lemma: If G is a simple graph & $\mathrm{diam}(G) \geq 3$ then $\mathrm{diam}(G^c) \leq 3$.



$\because$ $\mathrm{diam}(G) \geq 3$

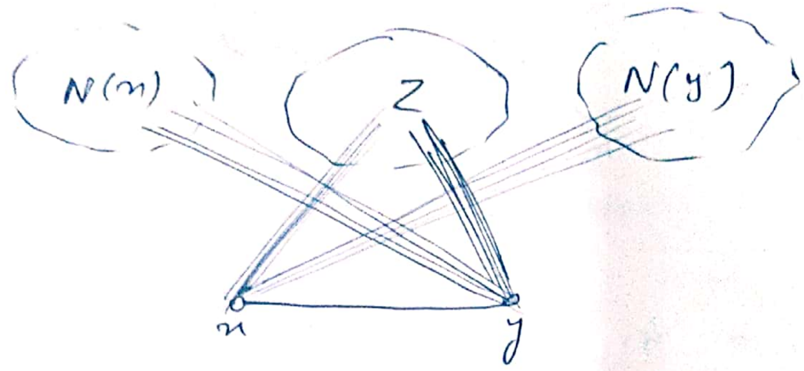
$\exists\ x, y \in V$

$\therefore$ $d_G(x,y) = 3$

s.t. $N(x) \cap N(y) = \phi$

Here $Z = V - \{x, y\} - N(x) - N(y)$

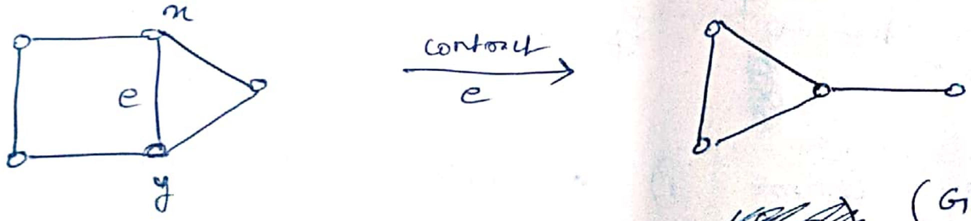
Taking G^c



In G^c it can be shown that for every pair of vertices $a, b \in V$ we have

$$d_{G^c}(a, b) \leq 3$$

Contraction of an edge



(G) (G·e)

$$\tau(G) = \tau(G-e) + \tau(G \cdot e)$$

where, τ fnx gives the no. of spanning trees in a graph

Matrix Tree Theorem :

Let $Q = D_{eg} - A$ where, $D_{eg} = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ is the degree mtx

& A is the adjacency matrix

then, $\tau(G) = C_{st}$ (cofactor)

$= (-1)^{s+t} \cdot \det(Q^*)$

where, Q^* is obtained by removing s^{th} row & t^{th} column from Q

Proof:

Claim 1 : If D is an arbitrary orientation of G & M is the incidence matrix of D then, $Q = MM^T$ where, $Q = Deg - A$

Proof of Claim 1 :

$$Q = \begin{array}{c|c} & e_1 \; e_2 \; \cdots \; e_m \\ \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \end{array} * \begin{array}{c|c} & v_1 \; v_2 \; \cdots \; v_n \\ \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

$\because Q = Deg - A \qquad Q_{ii} = d_G(v_i) - 0 = d_G(v_i)$

$\& \quad Q_{ij} = 0 - A_{ij} = -A_{ij} \quad (i \neq j)$

Arbitrary Orientation $\Rightarrow$ giving arbit directions to every edge in G giving a directed graph D.

It can be shown that the pdt $MM^T = Deg - A$

Claim 2 : If B is an $(n-1) \times (n-1)$ submatrix of M then $\det(B) = \pm 1$ if these $n-1$ edges form a spanning tree of G ; otherwise $\det(B) = 0$

Proof of Claim 2 : By induction

$n = 1$, by convention (vacuously true) $\det(B) = \pm 1$

Let it be true for $n-1$ & now we see for n

T be the spanning tree formed by "$n-1$" edges

$\exists$ two leaves x & y in T.

Out of the n vertices we take a sub-matrix of $(n-1) \times (n-1)$ then there will be atleast one row corresponding to a leaf. In that row we will have exactly one non zero entry & so finding the determinant along that row we get $(\pm 1) \times$ [det of $(n-2)\times(n-2)$ matrix]

By induction det for $(n-2) \times (n-2)$ will be 0 or ± 1 according as cycle or spanning tr.

So the $(n-1) \times (n-1)$ will also have similar det. Hence the claim is true ...

<u>Cauchy - Binnet Formula</u> :

$$\det(A \cdot B) = \sum_{S} \det(A_S) \cdot \det(B_S)$$

where, $A : p \times m$ & $B : m \times p$ $(m \geq p)$

& S is any p-set from $\{1, 2, 3, \ldots, m\}$

<u>Proof Contd...</u>

Let M^* be the matrix obtained from M by removing the r^{th} row.

$\therefore Q^* = M^* (M^*)^T$

If $m < n-1$, $\det(Q^*) = 0$ [$\because$ claim 2]

If $m \geq n-1$, $\det(Q^*) = \sum_{n-1} \det(M^*) \det((M^*)^T)$

$= \sum_{n-1} (\pm 1)^2 = \tau(G)$

[$\because$ 1 is contributed to sum if the n-1 edge set forms a spanning tree in G]

Algorithm for Spanning Tree : Prim's & Kruskal's
(Study Correctness proof)

Matching in Graphs :

A matching in a graph G is a subset M of edges of G such that no two edges share a common ~~ed~~ end point.
In other words, it is a subset of edges in which no two edges are adjacent i.e., an "independent" set of edges.

Defn : Let M be a matching in G & edge $uv \in M$. Then we say u & v are saturated by M.

A vertex v is said to be saturated by M if ∃ an edge in M whose one of the end-points is v.

Maximal matching is one in which no more edges can be added.

Maximum matching is one with max cardinality

Perfect Matching is one which saturates all vertices

Eg

$M = \{ab, cd, ef\}$ $M' = \{bc, de\}$

Given a matching M, an M-alternating path is a path which alternates btw. edges in M & edges not in M

M' - alternating path

M′ – augmenting path : an alternating path which does not ~~satisfies~~ saturate the end points of the path

Tutorial 2

8. $\sum_{v \in T} d(v) = 2(n-1)$

Let there be k- leaves in T ∴ $k \geq 2$

The contribution of each leaf in deg sum is unity

& that of max degree vertex is Δ &

that of other vertices is atleast 2.

$\therefore \sum d(v) \geq \Delta + k + (n-k-1)\cdot 2$

$\Rightarrow \cancel{2n} \cancel{-2} \geq \Delta + k + \cancel{2n} - 2k \cancel{-2}$

$\Rightarrow \underline{k \geq \Delta}$

11. By contradiction,

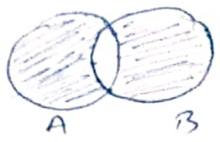
Let no two leaves have a common neighbour.

∴ For every leaf ∃ unique vertex adjacent to it with degree 3.

$\therefore \sum \deg(G) = 2(n-1) \geq k + 3k + 2(n-2k)$

$\Rightarrow 2(n-1) \geq 2n \quad (\Rightarrow\Leftarrow)$

Symmetric Difference : $A \Delta B = (A \cup B) - (A \cap B)$
$= (A/B) \cup (B/A)$



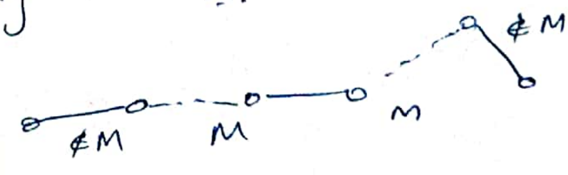
Lemma : If M & M' are two matchings of a graph G, then each ~~comp~~ component of subgraph $M \Delta M'$ is either a path or an even cycle.

Proof Let $F = M \Delta M'$ & $v \in V(F)$
then v has either one or two edges incident on it in F. $\therefore \deg_F(v) \leq 2$
So either it is part of a path or a even cycle [it cannot be an odd cycle as ~~as~~ v must have one edge from M & the other from M']

Theorem : A matching M of a graph G is maximum iff there is no M- augmented path in G.

Proof ($\Rightarrow$:) If $\exists$ an M-augmenting path then we get a matching M' st.

$|M'| \geq |M| + 1$.

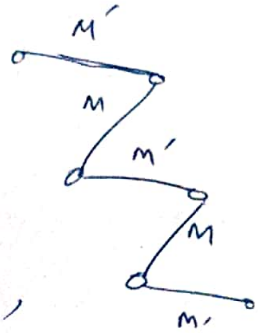


This contradicts that M is a maximum matching.

($\Leftarrow$) Let there be no M-aug path & let M' be a larger matching than M $[|M'|>|M|]$.

Tut 8

Let $F = M \Delta M'$ then by previous lemma, each component of F is either an even cycle or a path.



Since $|M'| > |M|$, there will be a component of F that contains an M augmenting path, which is a contradiction.

Hence M is maximum matching.

<u>Hall's Theorem</u>: Let G be a biportite graph with partitions X & Y. Then a matching M ~~satisfies~~ saturates X iff $[|X| \le |Y|]$

11. $|N(S)| \geq |S|$ for every $S \subseteq X$

Proof: ($\Rightarrow$:) Let $S \subseteq X$ then $\exists$ unique vertex for each member of S in Y in the matching M. $\therefore |N(S)| \geq |S|$ (There may be other edges from $x \in S$ to Y that are not included in M)

($\Leftarrow$:)

Let $|N(S)| \geq |S| \quad \forall \, S \subseteq X$.

Assume that M does not saturate X.

Let $u \in X$ be a vertex not saturated by M

[Here M is assumed to be a maximum matching.]

Define, $S = \{x \in X \mid x$ can be reached from u by an M- alternating path$\}$

$T = \{y \in Y \mid y$ can be reached from u by an M- alternating path$\}$

$u \in S$ [by defn.]

<u>claim</u> : M saturates T with $S - \{u\}$

$|T| = |S| - 1 \Rightarrow T \subseteq N(S)$

<u>Claim</u> : $T = N(S)$

<u>Proof</u> : Let $y = N(S) - T \quad \therefore \exists\, v \in S$ s.t. $vy \in E$

$\Rightarrow \exists$ an M- alternating path from u to v.

$\Rightarrow \exists$ an M- alternating path from u to y

$\Rightarrow y \in T \quad (\Rightarrow : \Leftarrow)$

$\therefore N(S) \subseteq T \quad \Rightarrow T = N(S)$

$\Rightarrow N(S) = T \Rightarrow |N(S)| = |T| = |S| - 1 < |S|$

which is a contradiction.

$\Rightarrow$ M must saturate X.

<u>Note</u> : The Hall's matching theorem is known as Marriage theorem when $|X| = |Y|$.

<u>Lemma</u> : Every k-regular bipartite graph $(k > 0)$ has a perfect matching

<u>Proof</u> : k-regular bip. $\Rightarrow |X| = |Y|$

Let $S \subseteq X$. For each vertex in S there are 'k' edges incident on it.

$\Rightarrow k.|S|$ edges.

These $k|S|$ edges are incident on ~~S~~ $N(S)$

& if every neighbour of S is unique

then there are at most $k|N(S)|$ edges

$\therefore$ $k|S| \leq k|N(S)|$ $\Rightarrow$ $|N(S)| \geq |S|$ $\forall S \subseteq X$

$\because$ Halls condition satisfied ~~perfect~~

$\Rightarrow$ $\exists$ matching that saturates X

$\Rightarrow$ $\exists$ perfect matching $[|X| = |Y|]$.

Min - Max Theorems

Konig - Egervary Theorem : If G is a bipartite graph without isolated vertices, then the size of a minimum vertex cover is equal to the size of the maximum matching

Proof : Notation - $|\text{min v.c.}| = \beta(G)$

$|\text{max matching}| = \alpha'(G)$

Let M be a maximum matching of G.

Then any VC of G must contain atleast one end point of every edge in M to cover atleast every edge in M.

$$\Rightarrow \beta(G) \geq \alpha'(G)$$

Now,

Let S be a minimum VC of G

Define

$$\left.\begin{array}{l} A = S \cap X \\ B = S \cap Y \end{array}\right\} \quad A \cup B = S$$

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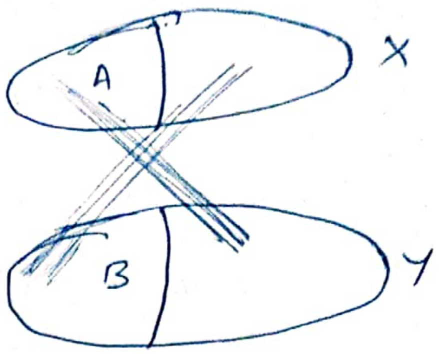
$\because$ G is a bipartite graph ~~every edge~~ & $A \cup B$ forms a vertex cover



$\Rightarrow$ Every edge must have atleast one end point in A or B.

Define the following induced subgraphs of G.

$H_1 = G[A \cup (Y - B)]$; $H_2 = G[B \cup (X - A)]$

Consider graph H_1. Let $P \subseteq A$.

Claim : $|N_{H_1}(P)| \geq |P|$

Proof by Contradiction : If $|N_{H_1}(P)| < |P|$ then by replacing the vertices P by $N_{H_1}(P)$ in set A we get a VC of G that has cardinality less than $|S|$ $(\Rightarrow\Leftarrow)$

$\therefore$ $|N_{H_1}(P)| \geq |P|$ $\Rightarrow \exists$ matching M_1 saturating A

Similarly, $\exists$ matching M_2 saturating B

Now, $M_1 \cup M_2$ is a matching of G s.t.

$M = M_1 \cup M_2$ & $|M| = \underline{|S| = \beta(G)}$

Also, $\alpha'(G) \geq |M|$ $\Rightarrow$ $\underline{\alpha'(G) \geq \beta(G)}$

Combining the two results we get,

$\underline{\alpha'(G) = \beta(G)}$ (proved)

Corollary: $|\text{Max. Independent Set}| = \alpha(G)$

$|\text{Min. Edge Cover}| = \beta'(G)$

then, i) $\alpha(G) + \beta(G) = n$ } G has no isolated vertices $n = |V|$

ii) $\alpha'(G) + \beta'(G) = n$

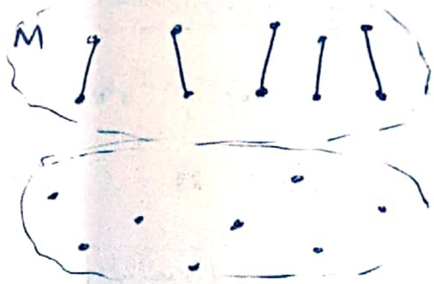
Proof i) Let S be indep. set then prove that $V - S$ is V.C. & $\beta(G) \leq n - \alpha(G)$

Then conversely prove $\alpha(G) \geq n - \beta(G)$

$\therefore \alpha(G) + \beta(G) = n$

ii) Let M be a max matching of G.

Now, for each v of G, ~~pick an edge~~ ~~incident on v~~. which is not in M, v must be adjacent to some vertex in M [as it cannot be isolated & it cannot be adjacent to any other vertex not in M, since it will contradict maximality of matching M]



So, for each unsaturated vertex v (by M) pick an edge incident on v to cover it. These edges together with M forms an edge cover of G with size $= n - 2\alpha' + \alpha' = n - \alpha'$

$\therefore \beta'(G) \leq n - \alpha'(G)$

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Next, we prove that $\beta'(G) \geq n - \alpha'(G)$

$\Rightarrow \alpha'(G) \geq n - \beta'(G)$.

Let L be a min edge cover of G & H be a component of L. Then it can be shown that H must be a star.

Collect one edge from each component of L. & let there be k components (stars)

$\therefore |L| = n - k \Rightarrow k = n - |L|$

These edges will form a matching M' of G

$\therefore \alpha'(G) \geq |M'|$

But, $|M'| = k = n - |L| = n - \beta'(G)$

$\therefore \alpha'(G) \geq n - \beta'(G)$

Hence, $\alpha'(G) + \beta'(G) = n$

<u>Tutte's Theorem</u>: For every $S \subseteq V(G)$,

$o(G-S) \leq |S| \iff G$ has a <u>perfect matching</u> (1-factor)

where,

$o(G-S)$ = no. of odd components in $G-S$.

odd component $\Rightarrow$ component have odd no. of vertices

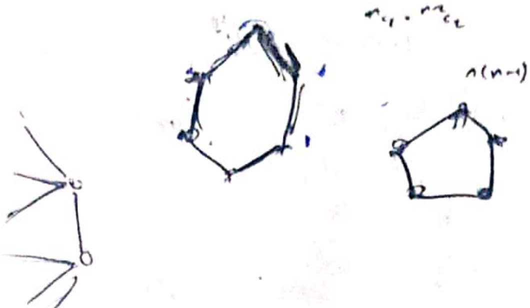
<u>Factor</u>: It is a spanning subgraph of a graph

<u>k-factor</u>: It is a spanning k-regular subgraph of a graph.

$\Rightarrow$ 1-factor = perfect matching.

C_6 $(2n-1)\cdot(2n-3)\cdots(1)$

C_n



Proof: $(\Leftarrow)$ Let G has a 1-factor then, each vertex of the components of $G-S$ has to be matched with another vertex. This implies that $o(G-S) \leq |S|$

$(\Rightarrow)$ Since introducing new edges in G will not increase $o(G-S)$, we consider a graph G' s.t. $o(G'-S) \leq |S|$ $\forall S \subseteq V$

Let, G' has no 1-factor & adding one (any) more edge in G' we get a 1-factor

Claim: G' has a 1-factor

Lovasz's proof: let $U = \{v \in V(G') \mid d(v) = n-1\}$

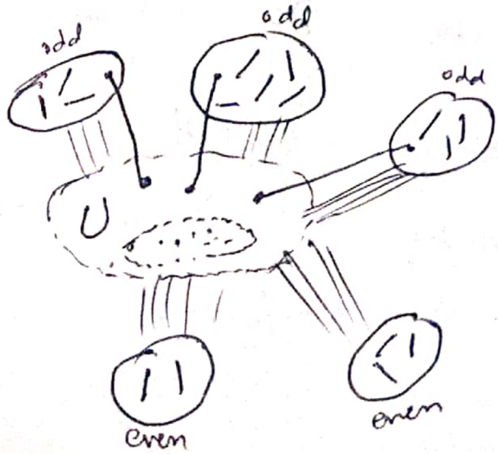
where, $n = |V(G')| = |V(G)|$

Case-1: $G'-U$ is the union of disjoint complete graphs. $- H_1, H_2, \ldots, H_k$

if $|H_i| = $ odd, $M_i = $ max matching of H_i

then, $V(H_i - M_i) = \{v\}$

Since each vertex of U is adjacent to every vertex of the graph $\exists\, v' \in U$ st. $vv' \in E(G')$



Let $M = \bigcup M_i \cup \{ v v' \mid v \in H_i \text{ \& } |V(H_i)| \text{ is odd} \}$

$U' = U - V(M)$

Then since, $o(G-S) \leq |S| \ \forall \ S \subseteq V$ in G'

for $S = \phi$ it should hold. [But it doesn't if $|U'|$ is odd]

$\Rightarrow |U'| = \text{even}$;

$U' \subseteq U \Rightarrow U'$ is a clique.

$M_{U'} \rightarrow$ maximum matching of $G'[U']$.

$\therefore M \cup M_{U'}$ is a 1-factor of G'

Case 2: $G' - U$ is not union of disjoint complete subgraphs.

$\therefore \exists$ a component H of $G' - U$ which is not complete.

$\Rightarrow \exists \ x, z \in H$ s.t. $xz \notin E(G')$

Moreover $\exists \ y \in V(H)$ s.t. $xy, yz \in E(G')$

[ie, $\exists$ a common neighbour of x & z]

Also, $y \notin U$

Now, ∵ $y \notin U \Rightarrow d_{G'}(y) < n-1$

$\Rightarrow \exists\, w \in V(G')$ st $wy \notin E(G')$

According to assumption adding either wy or xz to G' creates a 1-factor in G'.

Let $G' + xz \longrightarrow M_1$ (1-factor)

$G' + yw \longrightarrow M_2$ (1-factor)

∵ M_1 & M_2 are perfect matchings their symmetric diff can not be an odd path.

$\Rightarrow F = M_1 \Delta M_2 \Rightarrow$ even cycle [Prev. theorem]

More specifically the components of F are even cycles.

Let C be an even cycle of F containing edge xz [$xz \in M_1$, $xz \notin M_2 \Rightarrow xz \in F$]

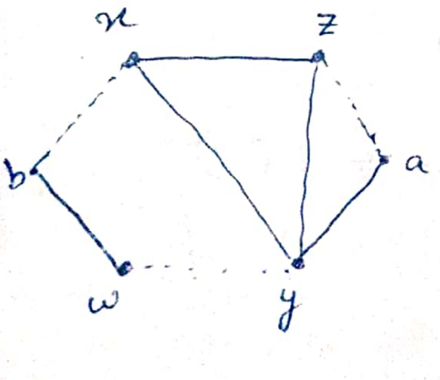
Also, i) let $yw \notin C \Rightarrow yw \in C'$ [Also even cycle]

Define M = (The edges of M_2 in C) ∪ (The edges of M_1 in C') ∪ (Remaining edges of M_1)

∴ $|M| = |M_1| = |M_2|$ & $xz, yw \notin M$

⇒ M is a 1-factor of G'

ii) let, C contain both xz & yw



Let M = xy ∪ (edges from M_2 while traversing y to x) ∪ (the edges of M_1 from w to x)

xy, az, bw

$\therefore |M| = |M_1| = |M_2|$

$\Rightarrow M$ is a a 1-factor of G'

This completes the proof of claim.

$\therefore G'$ has 1-factor $(\Leftrightarrow : \Leftarrow)$

by contradiction, G has a 1-factor

<u>Cycles</u>

<u>Eulerian Circuit</u> : A closed trail containing all the edges of the graph.

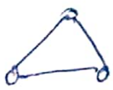
<u>Theorem</u> : A graph G contains an Eulerian circuit iff $d_G(v) = \text{even}$ $\forall$ $v \in V(G)$

<u>Proof</u> : $(\Rightarrow :)$ Let G be Eulerian then G must have a closed trail, containing all edges.

Let $v \in V(G)$ be arbitrary. Since for every edge incident on v there must be another edge incident on v. This is true since G is Eulerian. $\Rightarrow d_G(v) = \text{even}$

$(:\Leftarrow)$ We prove this by induction on m (no. of edges)

<u>$m=3$</u>



$\Rightarrow$ It has Eulerian circuit

Let G has m edges. Since $d_G(v) \geq 2$

$\Rightarrow \exists$ a cycle C in G.

Let $G' = G - E(G)$ & $|E(G')| < |E(G)| = m$

$d_{G'}(v) = \text{even} \quad \forall \ v \in V(G') = V(G)$

By induction, G' has an Eulerian ckt. C'

<u>Claim</u>: $C' \cup E(C)$ is an Eulerian ckt. of G.